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Abstract We consider the equation div $\mathbb{Y} = f$, with f a zero average function on the torus $\mathbb{T}^d$. In their seminal paper [5], Bourgain and Brezis proved the existence of a solution $\mathbb{Y} \in W^{1,d} \cap L^\infty$ for a datum $f \in L^d$. We extend their result to the critical Sobolev spaces $W^{s,p}$ with $(s+1)p = d$ and $p \geq 2$. More generally, we prove a similar result in the scale of Triebel-Lizorkin spaces. We also consider the equation div $\mathbb{Y} = f$ in a bounded domain Ω subject to zero Dirichlet boundary condition.

Keywords Divergence equation · Triebel-Lizorkin spaces · critical embedding

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1 Introduction

This paper is devoted to the regularity of solutions of the equation

$$\operatorname{div} \mathbb{Y} = f, \qquad \text{where} \int f = 0. \tag{1.1}$$

Here, the scalar function f and the vector field $\mathbb{Y}$ are defined on the d-dimensional torus[1] $\mathbb{T}^d$ or in a smooth bounded domain in $\mathbb{R}^d$.

Standard regularity theory asserts that $\mathbb{Y}$ gains one derivative with respect to f. For example, when $f \in L^p$, $1 < p < \infty$, one can pick $\mathbb{Y} \in W^{1,p}$. [2] In the limiting cases where $f \in L^1$, respectively $f \in L^\infty$, it is not always possible to pick $\mathbb{Y} \in W^{1,1}$ [5], respectively $\mathbb{Y} \in W^{1,\infty}$ [18]; see also [8, 13].

In their seminal paper, Bourgain and Brezis [5] discovered another limiting situation: the case where $f \in L^d$. For such f, (1.1) has a solution $\mathbb{Y} \in W^{1,d}$, so that $\mathbb{Y}$ "almost" belongs to L^∞. It turns out that (1.1) does have a solution in L^∞. This was noted in [5, Proposition 1]. In principle, there is no reason to have a solution in both L^∞ and $W^{1,d}$. Bourgain and Brezis [5] obtained the existence of a solution of (1.1) in $L^\infty \cap W^{1,d}$ for a datum $f \in L^d$. The proof is a real *tour de force*: the construction of $\mathbb{Y}$ is highly nontrivial and the proof of the fact that $\mathbb{Y}$ has the desired regularity is extremely involved. In the special case $p = d = 2$, existence of $\mathbb{Y} \in L^\infty \cap W^{1,2}$ can be established via a simpler duality argument [5, Section 4]. More generally, when $f \in W^{d/2-1,2}$, existence of a solution $\mathbb{Y} \in L^\infty \cap W^{d/2,2}$ of (1.1) can be obtained by a similar strategy [12, 14]. However, no simple proof of existence is known when $p \neq 2$.

In this paper, we investigate the regularity properties of the Bourgain-Brezis field $\mathbb{Y}$ in other function spaces. We consider regularity in Sobolev-Slobodeskii spaces $W^{s,p}$ and more generally, in Triebel-Lizorkin spaces $F^{s,p}_q$, focusing on the limiting situation where $(s+1)p = d$. In this case, if $f \in F^{s,p}_q$, we expect a solution $\mathbb{Y} \in F^{s+1,p}_q$, and the latter space is "close" to L^∞, thanks to the condition $(s+1)p = d$.

Our main result is

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[1] We identify $\mathbb{T}^d$ with $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.

[2] This follows from elliptic regularity theory: it suffices to let $\mathbb{Y} = \nabla u$, where u is an appropriate solution of $\Delta u = f$.

Theorem 1.1 *Let $p \ge 2$, $q \in [2,p]$, $d \ge 2$ and $s > -\dfrac{1}{2}$. We assume that*

$$(s+1)p = d. \tag{1.2}$$

Let $f \in F^{s,p}_q(\mathbb{T}^d)$ satisfy the compatibility condition $\int_{\mathbb{T}^d} f = 0$. Then (1.1) has a solution $\mathbb{Y} \in L^\infty(\mathbb{T}^d) \cap F^{s+1,p}_q(\mathbb{T}^d)$ such that

$$\|\mathbb{Y}\|_{L^\infty(\mathbb{T}^d)} + \|\mathbb{Y}\|_{F^{s+1,p}_q(\mathbb{T}^d)} \le C\|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{1.3}$$

The result of Bourgain and Brezis corresponds to $s = 0$, $p = d$, $q = 2$. In particular, Theorem 1.1 contains the following regularity result in the scale of Sobolev-Slobodeskii spaces:

Corollary 1.2 *Let $p \ge 2$ and $d \ge 2$. We assume that (1.2) holds and that $s > -\dfrac{1}{2}$. Let $f \in W^{s,p}(\mathbb{T}^d)$ be such that $\int_{\mathbb{T}^d} f = 0$. Then (1.1) has a solution $\mathbb{Y} \in L^\infty(\mathbb{T}^d) \cap W^{s+1,p}(\mathbb{T}^d)$ such that*

$$\|\mathbb{Y}\|_{L^\infty(\mathbb{T}^d)} + \|\mathbb{Y}\|_{W^{s+1,p}(\mathbb{T}^d)} \le C\|f\|_{W^{s,p}(\mathbb{T}^d)}.$$

Our proof follows the main lines of the one of Bourgain and Brezis in [5]. In particular, as in [5], we make use of a one-sided inequality, due to Rubio de Francia [20]; see Theorem 5.1 below. This inequality requires $p \ge 2$; this is why the condition $p \ge 2$ appears in both Theorem 1.1 and Corollary 1.2.[3] Whether the assumption $p \ge 2$ can be removed is a challenging question. Observe also that we do not obtain the conclusions of Theorem 1.1 and Corollary 1.2 when $s \le -\dfrac{1}{2}$. This is probably due to our method.

The regularity result concerning the divergence equation was subsequently extended by Bourgain and Brezis [6] to more general Hodge systems. It is plausible that a version of Theorem 1.1 still holds for these systems. We will return to this question in a subsequent work.

Our paper is organized as follows. In Section 2, we recall the definition of the Triebel-Lizorkin spaces. In Section 3, we describe the Bourgain-Brezis construction, give the main steps of the proof of Theorem 1.1 and state the main estimates. In Sections 4 and 5 we collect the harmonic analysis background used in the proof of Theorem 1.1; the estimates we prove in Section 5 are crucial in the proof of Theorem 1.1. In Section 6, we establish the validity of the main estimates stated in Section 3. In Section 7, we discuss the solvability of (1.1) in bounded domains. More specifically, we prove the following

Theorem 1.3 *Let $p \ge 2$, $q \in [2,p]$, $d \ge 2$ and $s > -\dfrac{1}{2}$ satisfy (1.2). Let Ω be a smooth bounded domain in $\mathbb{R}^d$. Let $f \in F^{s,p}_q(\Omega)$ satisfy the compatibility condition $\int_\Omega f = 0$.[4]*

Then there exists $\mathbb{Y} \in F^{s+1,p}_q(\Omega) \cap L^\infty(\Omega)$ such that $\begin{cases} div\ \mathbb{Y} = f & in\ \Omega \\ tr\ \mathbb{Y} = 0 & on\ \partial\Omega \end{cases}$. Moreover, we can choose $\mathbb{Y}$ such that

$$\|\mathbb{Y}\|_{L^\infty(\Omega)} + \|\mathbb{Y}\|_{F^{s+1,p}_q(\Omega)} \le C\|f\|_{F^{s,p}_q(\Omega)}.$$

A final appendix gathers the proofs of some elementary estimates involving trigonometric polynomials used in the proof of Theorem 1.1.

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2 Projections and Triebel-Lizorkin spaces

We denote by $\mathscr{P}(\mathbb{T}^d)$ the space of trigonometric polynomials.

If Δ is a subset of $\mathbb{Z}^d$, then we denote by $\mathbb{P}_\Delta$ the projection on the Fourier coefficients in Δ:

$$\mathbb{P}_\Delta(f)(x) := \sum_{n\in\Delta} \widehat{f}(n)e^{\imath n\cdot x}.$$

This projection makes sense if either Δ is finite and $f \in \mathscr{D}'(\mathbb{T}^d)$, or Δ is arbitrary and $f \in \mathscr{P}(\mathbb{T}^d)$.

Of special importance in the theory of function spaces are the projections on dyadic sets. These sets are finite unions of intervals[5] and are defined as follows:

$$\Delta^d_0 = \{0\},\ \Delta^d_j := [2^{j-1} \le |n| < 2^j],\ \forall\, j \ge 1;$$

here, we work with the norm $|n| = \max\{|n_j|; j \in [\![1,d]\!]\}$.[6]

[3] The condition $p \ge 2$ amounts to $s \le d/2 - 1$.

[4] When $s < 0$, this condition has to be suitably interpreted, see Section 7 below.

[5] An interval in $\mathbb{Z}^d$ is a Cartesian product of intervals in $\mathbb{Z}$.

[6] The notation $[\![a,b]\!]$, with $a, b \in \mathbb{Z}$, denotes the set $\{n \in \mathbb{Z}; a \le n \le b\}$.

One of the possible equivalent definitions of the Triebel-Lizorkin spaces is [22, Theorem 3.5.3]

$$F_q^{s,p}(\mathbb{T}^d) = \left\{ f \in \mathscr{D}'(\mathbb{T}^d) : \|f\|_{F_q^{s,p}(\mathbb{T}^d)} = \left\| \left\| \left(2^{js} \mathbb{P}_{\Delta_j^d}(f)(x) \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} < \infty \right\}. \tag{2.1}$$

The scale of Triebel-Lizorkin spaces contains in particular most of the Sobolev-Slobodeskii spaces: when $1 < p < \infty$, we have

$$W^{s,p}(\mathbb{T}^d) = \begin{cases} F_2^{s,p}(\mathbb{T}^d), & \text{if } s \in \mathbb{N} \\ F_p^{s,p}(\mathbb{T}^d), & \text{if } s \notin \mathbb{N} \end{cases}.$$

In particular, when $s = 0$, the above and (2.1) amount to the square function theorem

$$\|f\|_{L^p} \sim \left\| \left\| \left(\mathbb{P}_{\Delta_j^d}(f)(x) \right) \right\|_{l^2(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)}, \; 1 < p < \infty. \tag{2.2}$$

A word about the condition $\int_{\mathbb{T}^d} f = 0$, which appears in (1.1): when s in Theorem 1.1 is positive, we have $f \in L^1(\mathbb{T}^d)$, so that the condition $\int_{\mathbb{T}^d} f = 0$ makes sense. For arbitrary s, this condition has to be understood as

$$\widehat{f}(0) = 0. \tag{2.3}$$

3 Proof of Theorem 1.1

It suffices to establish (1.3) when $f \in \mathscr{P}(\mathbb{T}^d)$. The general case is obtained by density. We use the notations which will be introduced in (3.6)-(3.8) below. Since $\widehat{f}(0) = 0$, we have $f = f^1 + \cdots + f^d$, with $f^k = \mathbb{P}_{\mathbb{B}^k}(f)$.
Theorem 1.1 will be a consequence of

Proposition 3.1 *There exist $\delta_0 > 0$ and $\alpha > 0$ such that, for every $f \in \mathscr{P}(\mathbb{T}^d)$ satisfying $\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \delta_0$, there exist functions $\mathbb{Y}_1, \ldots, \mathbb{Y}_d : \mathbb{T}^d \to \mathbb{C}$ such that*

$$\|\mathbb{Y}_k\|_{L^\infty(\mathbb{T}^d)} \le 1, \; \|\mathbb{Y}_k\|_{F_q^{s+1,p}(\mathbb{T}^d)} \le 1, \;\; \textit{for } k \in [\![1,d]\!], \tag{3.1}$$

and

$$\|\partial_k \mathbb{Y}_k - f^k\|_{F_q^{s,p}(\mathbb{T}^d)} \le \|f\|^{1+\alpha}_{F_q^{s,p}(\mathbb{T}^d)}, \;\; \textit{for } k \in [\![1,d]\!]. \tag{3.2}$$

From Proposition 3.1, we get at once

Corollary 3.2 *There exist $\delta_0 > 0$ and $\alpha > 0$ such that, for every $0 < \delta \le \delta_0$ and every $g \in F_q^{s,p}(\mathbb{T}^d)$ satisfying* (2.3)*, there exists $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_d) \in F_q^{s+1,p}(\mathbb{T}^d)$ such that*

$$\|\mathbb{X}_k\|_{L^\infty(\mathbb{T}^d)} \le \frac{\|g\|_{F_q^{s,p}(\mathbb{T}^d)}}{\delta}, \;\; \|\mathbb{X}_k\|_{F_q^{s+1,p}(\mathbb{T}^d)} \le \frac{\|g\|_{F_q^{s,p}(\mathbb{T}^d)}}{\delta}, \;\; \textit{for } k \in [\![1,d]\!], \tag{3.3}$$

and

$$\|\operatorname{div}\mathbb{X} - g\|_{F_q^{s,p}(\mathbb{T}^d)} \le d\delta^\alpha \|g\|_{F_q^{s,p}(\mathbb{T}^d)}. \tag{3.4}$$

Proof of Theorem 1.1 using Corollary 3.2. Let $f \in F_q^{s,p}(\mathbb{T}^d)$. We set $g_0 := f$ and $\delta := \min\left((2d)^{-1/\alpha}, \delta_0\right)$. By Corollary 3.2, there exists $\mathbb{X}^0 \in F_q^{s+1,p}(\mathbb{T}^d)$ such that

$$\|\mathbb{X}_k^0\|_{L^\infty(\mathbb{T}^d)\cap F_q^{s+1,p}(\mathbb{T}^d)} \le \frac{\|g_0\|_{F_q^{s,p}(\mathbb{T}^d)}}{\delta}, \text{ for } k \in [\![1,d]\!], \; \|\operatorname{div}\mathbb{X}^0 - g_0\|_{F_q^{s,p}(\mathbb{T}^d)} \le d\delta^\alpha \|g_0\|_{F_q^{s,p}(\mathbb{T}^d)}.$$

More generally, assume that $g_0, \ldots, g_l$ have been defined, as well as the corresponding vector fields $\mathbb{X}^0, \ldots, \mathbb{X}^l$ provided by Corollary 3.2. Thus we have

$$\|\mathbb{X}_k^l\|_{L^\infty(\mathbb{T}^d)\cap F_q^{s+1,p}(\mathbb{T}^d)} \le \frac{\|g_l\|_{F_q^{s,p}(\mathbb{T}^d)}}{\delta}, \text{ for } k \in [\![1,d]\!], \; \|\operatorname{div}\mathbb{X}^l - g_l\|_{F_q^{s,p}(\mathbb{T}^d)} \le d\delta^\alpha \|g_l\|_{F_q^{s,p}(\mathbb{T}^d)}. \tag{3.5}$$

We then set $g_{l+1} := g_l - \operatorname{div}\mathbb{X}^l$ and let $\mathbb{X}^{l+1} \in F_q^{s+1,p}(\mathbb{T}^d)$ satisfy (3.3) and (3.4) with g replaced by g_{l+1}.
Combining (3.5) with our choice of δ, we find that

$$\|g_{l+1}\|_{F_q^{s,p}(\mathbb{T}^d)} \le d\delta^\alpha \|g_l\|_{F_q^{s,p}(\mathbb{T}^d)} \le \ldots \le (d\delta^\alpha)^{l+1} \|g_0\|_{F_q^{s,p}(\mathbb{T}^d)} \le \frac{1}{2^{l+1}} \|g_0\|_{F_q^{s,p}(\mathbb{T}^d)}.$$

It then follows that

$$\|\mathbb{X}_k^l\|_{L^\infty(\mathbb{T}^d)\cap F_q^{s+1,p}(\mathbb{T}^d)} \le \frac{\|g_0\|_{F_q^{s,p}(\mathbb{T}^d)}}{2^l \delta}, \text{ for } k \in [\![1,d]\!].$$

We can thus define $\mathbb{X} := \sum_l \mathbb{X}^l$, which satisfies $\|\mathbb{X}\|_{L^\infty(\mathbb{T}^d)\cap F_q^{s+1,p}(\mathbb{T}^d)} \le C\|f\|_{F_q^{s,p}(\mathbb{T}^d)}$. Moreover, the fact that $\|g_l\|_{F_q^{s,p}(\mathbb{T}^d)}$ goes to 0 when $l \to +\infty$ implies that $\operatorname{div}\mathbb{X} = f$. This completes the proof of Theorem 1.1. □

It remains to explain the

Proof of Proposition 3.1. We write

$$\mathbb{Z}^d \setminus \{0\} = \bigcup_{j\geq 0} \left(\mathbb{B}_j^1 \cup \dots \mathbb{B}_j^d\right), \tag{3.6}$$

with

$$\mathbb{B}_j^1 := \{n \in \Delta_j^d : 2^{j-1} \leq |n_1| < 2^j\}, \mathbb{B}_j^2 := \{n \in \Delta_j^d : 2^{j-1} \leq |n_2| < 2^j\} \setminus \mathbb{B}_j^1, \dots, \mathbb{B}_j^d := \Delta_j^d \setminus \left(\bigcup_{k=1}^{d-1} \mathbb{B}_j^k\right). \tag{3.7}$$

Hence, for each $j \geq 0$, $\Delta_j^d = \bigcup_{k=1}^{d} \mathbb{B}_j^k$, the union being disjoint. We also let

$$\mathbb{B}^k = \bigcup_{j\geq 0} \mathbb{B}_j^k, \ \text{ for } k \in [\![1,d]\!], \tag{3.8}$$

and

$$\mathbb{B} = \mathbb{B}^1 \cap (\mathbb{N} \times \mathbb{Z}^{d-1}),\ \mathbb{B}_j = \mathbb{B}_j^1 \cap (\mathbb{N} \times \mathbb{Z}^{d-1}).$$

We describe the construction of $\mathbb{Y}_1$ which only involves $\mathbb{B}^1$, $\mathbb{B}_j^1$ and $f^1 = \mathbb{P}_{\mathbb{B}^1}(f)$. The component $\mathbb{Y}_k$, with $k \in [\![2,d]\!]$, is built in the same way from $f^k = \mathbb{P}_{\mathbb{B}^k}(f)$. We note that f^1 is a trigonometric polynomial, and that supp $\widehat{f^1} \subset \mathbb{B}^1$. We also note that

$$\begin{aligned}\|\mathbb{P}_{\mathbb{B}} f^1\|_{F_q^{s,p}(\mathbb{T}^d)} &= \left\|\left\|\left(2^{sj}\mathbb{P}_{\Delta_j^d}\mathbb{P}_{\mathbb{B}} f^1(x)\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)} = \left\|\left\|\left(2^{sj}\mathbb{P}_{\mathbb{B}_j}\mathbb{P}_{\Delta_j^d} f(x)\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)}\\ &\leq C\left\|\left\|\left(2^{sj}\mathbb{P}_{\Delta_j^d} f(x)\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)} = C\|f\|_{F_q^{s,p}(\mathbb{T}^d)},\end{aligned}$$

the above inequality being a consequence of Theorem 4.9. It follows that it suffices to prove Proposition 3.1 when f is replaced by $\mathbb{P}_{\mathbb{B}} f^1$. Therefore, we assume, in the sequel, that f is a trigonometric polynomial such that supp $\widehat{f} \subset \mathbb{B}$.

Following the strategy in [5], we divide the proof of Proposition 3.1 into five steps.

Step 1. *Estimates for the naive solution of* (1.1). When supp $\widehat{f} \subset \mathbb{B}$, an exact solution of div $\mathbb{X} = f$ is given by $\mathbb{X} = (F, 0, \dots, 0)$, where

$$F = \sum_{n\in\mathbb{B}} \frac{1}{\imath n_1}\widehat{f}(n)e^{\imath n\cdot x}. \tag{3.9}$$

The field $\mathbb{Y}_1$ will be constructed by modifying $\mathbb{X}$.

The purpose of Step 1 is to collect some estimates involving F and the projections of f and F, defined by

$$f_j := \sum_{n\in\mathbb{B}_j} \widehat{f}(n)e^{\imath n\cdot x} = \mathbb{P}_{\mathbb{B}_j} f = \mathbb{P}_{\Delta_j^d} f,\ \ F_j := \sum_{n\in\mathbb{B}_j} \frac{1}{\imath n_1}\widehat{f}(n)e^{\imath n\cdot x} = \mathbb{P}_{\mathbb{B}_j} F = \mathbb{P}_{\Delta_j^d} F. \tag{3.10}$$

More specifically, we will establish the following

Lemma 3.3 *We have*

$$\|F\|_{F_q^{s+1,p}(\mathbb{T}^d)} \leq C(s,p,q)\|f\|_{F_q^{s,p}(\mathbb{T}^d)}, \tag{3.11}$$

$$\left\|F_j\right\|_{L^\infty(\mathbb{T}^d)} \leq C(s,p,q)\left\|f_j\right\|_{F_q^{s,p}(\mathbb{T}^d)}, \tag{3.12}$$

$$\left\|\nabla F_j\right\|_{L^\infty(\mathbb{T}^d)} \leq C(s,p,q)2^j\left\|f_j\right\|_{F_q^{s,p}(\mathbb{T}^d)}, \tag{3.13}$$

and

$$\|f_j\|_{F_q^{s,p}(\mathbb{T}^d)} \leq C(s,p,q)\|f\|_{F_q^{s,p}(\mathbb{T}^d)}. \tag{3.14}$$

Step 2. *Construction of a good function H_j dominating $|F_j|$.* Let $\varepsilon = 2^{-\ell}$ with $\ell \in \mathbb{N}$ to be determined later. For $j > \ell$ we define $k(j) := 1/\varepsilon = 2^\ell$ and we decompose $\mathbb{B}_j$ into disjoint vertical strips $\mathbb{B}_{j,r}$, $r \in [\![1,k(j)]\!]$. More specifically, each $\mathbb{B}_{j,r}$ is of the form

$$\mathbb{B}_{j,r} = \left([\![a_{j,r}, b_{j,r}-1]\!] \times \mathbb{Z}^{d-1}\right) \cap \mathbb{B}_j,\ \text{ with } b_{j,r} - a_{j,r} = \varepsilon 2^{j-1} := l(j).$$

Following [5], we set

$$G_j(x) = \sum_{1\leq r\leq 1/\varepsilon}\left|\sum_{n\in\mathbb{B}_{j,r}} \frac{1}{n_1}\widehat{f}(n)e^{\imath n\cdot x}\right| \tag{3.15}$$

and

$$H_j(x) = 3^d G_j * (\mathbb{F}_{\varepsilon 2^j} \otimes \mathbb{F}_{2^j} \otimes \dots \otimes \mathbb{F}_{2^j}). \tag{3.16}$$

Here and after, we denote by $\mathbb{F}_N$ the Fejér kernel given by

$$\mathbb{F}_N(x) := \sum_{|n|\leq N} \frac{N-|n|}{N}e^{\imath nx} = \frac{1}{4N\pi}\frac{(\sin Nx)^2}{(\sin(x/2))^2}. \tag{3.17}$$

We set, for all $(i_1,...,i_d)\in\mathbb{N}^d$,

$$\mathbb{F}_{i_1}\otimes\ldots\otimes\mathbb{F}_{i_d}(x_1,\ldots,x_d):=\mathbb{F}_{i_1}(x_1)\ldots\mathbb{F}_{i_d}(x_d).$$

We clearly have, for all $x\in\mathbb{T}^d$,

$$|F_j(x)|\le G_j(x) \tag{3.18}$$

and also

$$\operatorname{supp}\widehat{H}_j\subset[|n|\le 2^j-1]. \tag{3.19}$$

A slightly more involved estimate, taken from [5], is that, for all $x\in\mathbb{T}^d$,

$$G_j(x)\le H_j(x); \tag{3.20}$$

this follows from Corollary 8.5 in the appendix.
Up to now, the functions G_j and H_j have been defined only for $j>\ell$. When $j\le\ell$ (that is, when $\varepsilon 2^{j-1}<1$), we do not split $\mathbb{B}_j$. Instead, we let

$$a_{j,1}=2^{j-1},\ b_{j,1}=2^j,\ G_j=|F_j| \tag{3.21}$$

and define H_j via

$$H_j=3^dG_j*(\mathbb{F}_{2^j}\otimes\mathbb{F}_{2^j}\otimes\cdots\otimes\mathbb{F}_{2^j}). \tag{3.22}$$

In this case, we set $k(j)=1$ and $l(j)=b_{j,1}-a_{j,1}=2^{j-1}$. It will follow form the proof of (3.20) that estimates (3.18), (3.21) and (3.22) remain valid.
The main purpose of Step 2 is to establish the following estimates satisfied by G_j and H_j.

Lemma 3.4 *We have*

$$\sum_{1\le r\le k(j)}\left\|x\mapsto\sum_{n\in\mathbb{B}_{j,r}}\widehat{f}(n)e^{\imath n\cdot x}\right\|^p_{L^p(\mathbb{T}^d)}\le C(p)\|f_j\|^p_{L^p(\mathbb{T}^d)}\le C(p)2^{-sjp}\|f\|^p_{F^{s,p}_q(\mathbb{T}^d)}, \tag{3.23}$$

$$\|G_j\|_{L^\infty(\mathbb{T}^d)}\le C(s,p,q)k(j)^{1-2/p}\left\|f_j\right\|_{F^{s,p}_q(\mathbb{T}^d)}, \tag{3.24}$$

and

$$\|\nabla G_j\|_{L^\infty(\mathbb{T}^d)}\le C(s,p,q)2^jk(j)^{1-2/p}\|f_j\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{3.25}$$

An immediate consequence of (3.14), (3.16), (3.24) and of $\|\mathbb{F}_N\|_{L^1(\mathbb{T})}=1$ is

$$\left\|H_j\right\|_{L^\infty(\mathbb{T}^d)}\le C(s,p,q)k(j)^{1-2/p}\|f_j\|_{F^{s,p}_q(\mathbb{T}^d)}\le C(s,p,q)k(j)^{1-2/p}\|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{3.26}$$

Similarly, we have

$$\|\nabla H_j\|_{L^\infty(\mathbb{T}^d)}\le 3^d\|\nabla G_j\|_{L^\infty(\mathbb{T}^d)}\le C(s,p,q)2^jk(j)^{1-2/p}\|f_j\|_{F^{s,p}_q(\mathbb{T}^d)}\le C(s,p,q)2^jk(j)^{1-2/p}\|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{3.27}$$

Step 3. *Construction of* $\mathbb{Y}_1$. Following [5], we let

$$\mathbb{Y}_1:=\sum_{j\ge 0}F_j\prod_{k>j}(1-H_k).$$

Set

$$K_j:=\sum_{k<j}F_k\prod_{k<m<j}(1-H_m)\ \text{ if } j>0,\ K_0:=0.$$

The next result will be an immediate consequence of Lemma 3.4.

Lemma 3.5 *One has*

$$\mathbb{Y}_1=\sum_{j\ge 0}F_j-\sum_{j\ge 0}H_jK_j \tag{3.28}$$

and

$$\partial_1\mathbb{Y}_1=\sum_{j\ge 0}f_j-\sum_{j\ge 0}\partial_1(H_jK_j). \tag{3.29}$$

Assume furthermore that $k(j)^{1-2/p}\|f\|_{F^{s,p}_q(\mathbb{T}^d)}\le\eta_0$ *for all* $j\in\mathbb{N}$, *where* $\eta_0>0$ *is a constant which will be determined in the proof. Then, for all* $x\in\mathbb{T}^d$,

$$|\mathbb{Y}_1(x)|\le 1, \tag{3.30}$$

$$\left|K_j(x)\right|\le 1,\ \forall j\in\mathbb{N} \tag{3.31}$$

and

$$0\le\left|F_j(x)\right|\le G_j(x)\le H_j(x)\le 1,\ \forall j\in\mathbb{N}. \tag{3.32}$$

In particular, Lemma 3.5 implies the L^∞ estimate in (3.1).

Step 4. *Proof of* (3.2). The proof relies on

Lemma 3.6 *There exist $\delta_0 > 0$, $\alpha > 0$ and $C = C(s,p,q) > 0$ such that, if $\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \delta_0$, then*

$$\left\| \sum_{j\ge 0} \partial_1(H_j K_j) \right\|_{F_q^{s,p}(\mathbb{T}^d)} \le C \, \|f\|^{1+\alpha}_{F_q^{s,p}(\mathbb{T}^d)}. \tag{3.33}$$

Step 5. *Proof of* (3.1) *completed: estimate of* $\|\nabla \mathbb{Y}_1\|_{F_q^{s,p}(\mathbb{T}^d)}$. This is achieved via the following

Lemma 3.7 *Let η_0 be as in Lemma 3.6. Then there exists a constant $\delta_0 >$ such that, if*

$$\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \min\left(\varepsilon^{1-2/p}\eta_0, \delta_0\right), \tag{3.34}$$

then $\|\nabla \mathbb{Y}_1\|_{F_q^{s,p}(\mathbb{T}^d)} \le 1$.

It is easy to see that Proposition 3.1 is a consequence of the five above steps. □

4 Kernels and multipliers

We will make use of the following classical kernels defined on $\mathbb{T}$:

1. The Fejèr kernel $\mathbb{F}_N$, given by (3.17).
2. The de la Vallée Poussin kernel $\mathbb{V}_N = 2\mathbb{F}_{2N} - \mathbb{F}_N$.

If $\varphi : \mathbb{Z}^d \to \mathbb{C}$ and f is a trigonometric polynomial on $\mathbb{T}^d$, we let

$$T_\varphi(f)(x) := \sum_{n\in\mathbb{Z}^d} \varphi(n)\widehat{f}(n)e^{\imath n\cdot x}, \ x \in \mathbb{T}^d.$$

Similarly, given, for any $j \in \mathbb{N}$, a function $\varphi_j : \mathbb{Z}^d \to \mathbb{C}$ and a trigonometric polynomial f_j, we set $\varphi = (\varphi_j)_{j\in\mathbb{N}}$, respectively $f = (f_j)_{j\in\mathbb{N}}$, and define T_φ by the formula

$$f \mapsto \left(x \mapsto \left(\sum_{n\in\mathbb{Z}^d} \varphi_j(n)\widehat{f}_j(n)e^{\imath n\cdot x} \right)_{j\in\mathbb{N}} \right). \tag{4.1}$$

More generally, the above formulae make sense if the trigonometric polynomial f has coefficients in a vector space over $\mathbb{C}$.

We start by recalling a classical estimate on Fourier multipliers.

Theorem 4.1 *[23, Section I.6.2] Let $k \in \mathscr{D}'(\mathbb{T}^d)$. Assume that*

1. $\varphi := \widehat{k} \in l^\infty(\mathbb{Z}^d)$.
2. $k \in L^1_{loc}\left(\mathbb{T}^d \setminus \{0\}\right)$.

Then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\left|\left|T_\varphi f\right|\right|_{L^p(\mathbb{T}^d)} \le C_p B(\varphi) ||f||_{L^p(\mathbb{T}^d)} \quad , \quad \forall f \in \mathscr{P}(\mathbb{T}^d).$$

Here,

$$B(\varphi) := \max\left\{ \left|\left|\varphi\right|\right|_{l^\infty(\mathbb{Z}^d)}, \sup_{x\in\mathbb{T}^d} \int_{|x-y|\ge 2|x|} |k(y-x) - k(y)| \, dy \right\}.$$

The next result that we quote is a vector-valued version of Theorem 4.1; see [1, Theorem 2]. In [1], Theorem 4.2 is stated in $\mathbb{R}^d$, but the proof is easily adapted to $\mathbb{T}^d$.

Theorem 4.2 *[1, Theorem 2] Let A, B be Banach spaces and let $k \in L^1(\mathbb{T}^d, \mathscr{L}(A,B))$. Define $T : L^p(\mathbb{T}^d, A) \to L^p(\mathbb{T}^d, B)$ through the formula*

$$Tf(x) = k * f(x) = \int k(x-y)f(y)\,dy, \ \forall f \in L^p(\mathbb{T}^d, A). \tag{4.2}$$

Fix $1 < p_0 < \infty$ and let

$$M \ge \int_{|y-x|\ge 2|x|} \|k(y-x) - k(y)\|_{\mathscr{L}(A,B)}\,dy, \ \forall x \in \mathbb{T}^d. \tag{4.3}$$

Then, for $1 < p < \infty$, the norm $\|T\|_{L^p \to L^p}$ of T as a linear continuous operator from $L^p(\mathbb{T}^d, A)$ into $L^p(\mathbb{T}^d, B)$ is controlled by a quantity depending solely on M, on p and on $\|T\|_{L^{p_0}\to L^{p_0}}$, but not on $\|k\|_{L^1(\mathbb{T}^d, \mathscr{L}(A,B))}$.

We next derive some consequences of Theorem 4.2. To start with, we consider the case where

i. $d = 1$.
ii. $A = B = l^q(\mathbb{N})$, with $1 < q < +\infty$.
iii. The operator Tf acts, on a sequence $f = (f_j)_{j\in\mathbb{N}}$ of functions in $L^1(\mathbb{T})$, through the formula

$$Tf(x) = \left(\mathbb{F}_{N_j} * f_j(x)\right)_{j\in\mathbb{N}}. \tag{4.4}$$

More specifically, we will be interested in the case where

$$N_j = 2^{k_j} \text{ for some integer } k_j \ge 0, \ \forall j \in \mathbb{N}. \tag{4.5}$$

Lemma 4.3 *Let $1 < p < \infty$ and $1 < q < \infty$. Then T defined by* (4.4) *and* (4.5) *is continuous from $L^p(\mathbb{T}; l^q(\mathbb{N}))$ into $L^p(\mathbb{T}; l^q(\mathbb{N}))$. In addition, we have*

$$\left\| \left\| \left(\mathbb{F}_{N_j} * f_j \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \le C(p,q) \left\| \left\| \left(f_j \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})}, \tag{4.6}$$

with $C(p,q)$ independent of k_j.

Proof By a standard limiting procedure, it suffices to prove (4.6) (with bounds independent of $J \in \mathbb{N}$) for the truncated operator, still denoted T, defined on $L^p\left(\mathbb{T}, l^q\left([\![0,J]\!]\right)\right)$ with values into $L^p\left(\mathbb{T}, l^q\left([\![0,J]\!]\right)\right)$ through the formula

$$Tf(x) = \left(\mathbb{F}_{N_j} * f_j(x) \right)_{j\in[\![0,J]\!]}, \ \forall f_j \in L^p(\mathbb{T}, \mathbb{C}), \ j \in [\![0,J]\!].$$

For this purpose, it suffices to check that the assumptions of Theorem 4.2 are satisfied with $p_0 = q$, and that the norm $\|T\|_{L^q\to L^q}$ as well as the right-hand side of (4.3) have upper bounds independent of k_j and J.

Let us start by computing $\|T\|_{L^q\to L^q}$. For each j, we have

$$\int_{\mathbb{T}} |\mathbb{F}_{N_j} * f_j|^q(x)\,dx \le \int_{\mathbb{T}} |f_j|^q(x)\,dx,$$

since $\left\|\mathbb{F}_{N_j}\right\|_{L^1(\mathbb{T})} = 1$. By taking the sum over j in the above inequality, we find that $\|T\|_{L^q\to L^q} \le 1$.

We next obtain a uniform estimate for the right-hand side of (4.3). We have

$$\|k(y-x) - k(y)\|_{\mathscr{L}(l^q(\mathbb{N}), l^q(\mathbb{N}))} = \sup_j |\mathbb{F}_{N_j}(y-x) - \mathbb{F}_{N_j}(y)| \le \sum_k |\mathbb{F}_{2^k}(y-x) - \mathbb{F}_{2^k}(y)|. \tag{4.7}$$

In view of (4.7), it suffices to establish the estimate

$$\sum_k \int_{|y-x|\ge 2|x|} |\mathbb{F}_{2^k}(y-x) - \mathbb{F}_{2^k}(y)|\,dy \le C, \ \forall x \in \mathbb{T}. \tag{4.8}$$

In order to prove (4.8), we rely, on the one hand, on Bernstein's integral inequality [9, Theorem D.2.1], which yields

$$\int_{\mathbb{T}} |\mathbb{F}_N(y-x) - \mathbb{F}_N(y)|\,dy \le CN|x|. \tag{4.9}$$

On the other hand, we have $\mathbb{F}_N(y) \le \dfrac{C}{Ny^2}$, and therefore

$$\int_{|x-y|\ge 2|x|} |\mathbb{F}_N(y-x) - \mathbb{F}_N(y)|\,dy \le \frac{C}{N} \int_{|y|\ge|x|} \frac{dy}{y^2} \le \frac{C}{N|x|}. \tag{4.10}$$

Let $x \in \mathbb{T} \setminus \{0\}$. If $|x| \ge 1$, then (4.10) implies that

$$\sum_k \int_{|x-y|\ge 2|x|} |\mathbb{F}_{2^k}(y-x) - \mathbb{F}_{2^k}(y)|\,dy \le C \sum_k \frac{1}{2^k|x|} \le C, \tag{4.11}$$

guaranteeing the validity of (4.8) for such x.
Assume next that $|x| < 1$. Let $k_0 \in \mathbb{N}$ be such that $2^{-k_0-1} \le |x| < 2^{-k_0}$. Thanks to (4.9) and (4.10), we have

$$\sum_k \int_{|x-y|\ge 2|x|} |\mathbb{F}_{2^k}(y-x) - \mathbb{F}_{2^k}(y)|\,dy = \sum_{k\le k_0} \ldots + \sum_{k>k_0} \ldots \le C|x| \sum_{k\le k_0} 2^k + \frac{C}{|x|} \sum_{k>k_0} \frac{1}{2^k} \le C,$$

which proves (4.8). This completes the proof of the lemma. □

Lemma 4.3 combined with a Fubini type argument[7] leads to

Lemma 4.4 *Let $1 < p < \infty$ and $1 < q < \infty$. Let*

$$T\left((f_j)_{j\in\mathbb{N}} \right) = \left(\left(\mathbb{F}_{2^{k_{1,j}}} \otimes \ldots \otimes \mathbb{F}_{2^{k_{d,j}}} \right) * f_j \right)_{j\in\mathbb{N}}, \ \forall f_j \in \mathscr{P}(\mathbb{T}^d), \ \forall j \in \mathbb{N}.$$

Then

$$\left\| \left\| T\left((f_j)_{j\in\mathbb{N}} \right) \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \le C(p,q) \left\| \left\| (f_j)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)}, \tag{4.12}$$

with $C(p,q)$ independent of $k_{1,j}, \ldots, k_{d,j}$.

Let us also recall the scalar and the vector-valued Marcinkiewicz Theorem on Fourier multipliers:

Theorem 4.5 *Let $\varphi : \mathbb{Z} \to \mathbb{C}$ and $1 < p < \infty$.*

[7] A similar argument is detailed in the proof of Corollary 4.6.

1. *[9, Theorem 8.2.1] The operator T_φ, initially defined for trigonometric polynomials $f \in \mathscr{P}(\mathbb{T}^d)$, can be extended to an L^p-bounded operator satisfying*
$$\|T_\varphi f\|_{L^p(\mathbb{T})} \le C(p)A(\varphi)\|f\|_{L^p(\mathbb{T})}, \tag{4.13}$$
provided
$$A(\varphi) := \max\left(\|\varphi\|_{l^\infty(\mathbb{Z})}, \sup_j \sum_{n\in\Delta_j^1} |\varphi(n+1)-\varphi(n)|\right) < \infty.$$
2. *[25, Proposition 2] Let X be a Banach space satisfying the* UMD *property. Then the conclusion of item 1. holds for X-valued maps. More specifically, if $A(\varphi) < \infty$, then the operator*
$$f \mapsto T_\varphi(f) = \sum_{n\in\mathbb{Z}} \varphi(n)\widehat{f}(n)e^{\imath nx},$$
initially defined for trigonometric polynomials f with coefficients in X, $f \in \mathscr{P}(\mathbb{T},X)$, can be extended to a linear operator on $L^p(\mathbb{T},X)$ satisfying (4.13).

For a discussion on the UMD property, see [11]. Some results related to item *2.* above and the UMD property can be found, e.g., in [10]. Of importance for us is that the space $l^q(\mathbb{N})$, with $1<q<\infty$, has the UMD property.

We will also need the following d-dimensional version of Theorem 4.5:

Corollary 4.6 *Let $1<p<\infty$. Let X be a Banach space with the* UMD *property, $\varphi_j : \mathbb{Z} \to \mathbb{C}, 1 \le j \le d$ and $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_d$. Then*
$$\|T_\varphi f\|_{L^p(\mathbb{T}^d,X)} \le C(p,X)A(\varphi_1)\ldots A(\varphi_d)\|f\|_{L^p(\mathbb{T}^d,X)},\ \forall f \in \mathscr{P}(\mathbb{T}^d),$$
where
$$A(\varphi_l) := \max\left(\|\varphi_l\|_{l^\infty(\mathbb{Z})}, \sup_j \sum_{n\in\Delta_j^1} |\varphi_l(n+1)-\varphi_l(n)|\right).$$

Proof If $g : \mathbb{T}^d \to X$, we denote by $g_{*x'}$ the map $x_1 \mapsto g(x_1,x')$. Observe that
$$T_\varphi f(x) = T_{\varphi_1}((T_\psi f)_{*x'})(x_1),$$
where $\psi(n_1,n_2,\ldots,n_d) = \varphi_2(n_2)\ldots\varphi_d(n_d)$. Theorem 4.5 and the Fubini theorem imply
$$\begin{aligned}\|T_\varphi f\|^p_{L^p(\mathbb{T}^d,X)} &= \int_{\mathbb{T}^{d-1}} \|T_{\varphi_1}((T_\psi f)_{*x'})\|^p_{L^p(\mathbb{T},X)}\,dx' \lesssim A(\varphi_1)^p \int_{\mathbb{T}^{d-1}} \|(T_\psi f)_{*x'}\|^p_{L^p(\mathbb{T},X)}\,dx'\\ &= A(\varphi_1)^p\|T_\psi f\|^p_{L^p(\mathbb{T}^d,X)} \lesssim \cdots \lesssim A(\varphi_1)^p\ldots A(\varphi_d)^p\|f\|^p_{L^p(\mathbb{T}^d,X)}.\end{aligned}$$
□

Here and in what follows, $\lesssim$ stands for $\le C$, with appropriate C.

We will often rely on the following special case of Corollary 4.6.

Corollary 4.7 *Let X be a Banach space with the* UMD *property and let $1<p<\infty$.*

Let $f \in \mathscr{P}(\mathbb{T}^d,X)$ be such that $\operatorname{supp} \widehat{f} \subset \prod_{j=1}^d [\![a_j,b_j]\!]$.

Let $\varphi_j : \mathbb{Z} \to \mathbb{R}$, $j \in [\![1,d]\!]$, be monotonic in the sets $[n_j>0]$ and $[n_j<0]$. Set $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_d$. Then
$$\|T_\varphi f\|_{L^p(\mathbb{T}^d,X)} \le C(p,X)\|\varphi_1\|_{l^\infty([\![a_1,b_1]\!])}\cdots\|\varphi_d\|_{l^\infty([\![a_d,b_d]\!])}\|f\|_{L^p(\mathbb{T}^d,X)}. \tag{4.14}$$

Proof Apply Corollary 4.6 to the functions $\varphi_j \mathbb{1}_{[\![a_j,b_j]\!]}$. □

Vector-valued Fourier multiplier are valid beyond multipliers φ of the form $\varphi_1 \otimes \ldots \otimes \varphi_d$. Here is a special case we will rely on in the sequel. The next result is a straightforward consequence of [25, Proposition 2].

Lemma 4.8 *Let $1<p<\infty$ and $1<q<\infty$. Let $(P_j)_{j\in\mathbb{N}}$ be a sequence of trigonometric polynomials. Then*
$$\left\|\left\|\left(\partial_1 \mathbb{P}_{\Delta_j^d}(P_j)\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)} \le C(p,q)\left\|\left\|\left(2^j \mathbb{P}_{\Delta_j^d}(P_j)\right)\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)}. \tag{4.15}$$

We will also make use of the following vector-valued version of the Riesz inequality.

Theorem 4.9 *[19,3] Let $1<p<\infty$ and $1<q<\infty$. Let $(I_j)_j$ be an arbitrary sequence of intervals in $\mathbb{Z}^d$. For all $j \in \mathbb{N}$, let $f_j \in \mathscr{P}(\mathbb{T}^d)$. Then*
$$\left\|\left\|\left(\mathbb{P}_{I_j} f_j\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)} \le C(p,q)\left\|\left\|(f_j)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)}.$$
Here, $C(p,q)$ is independent of the choice of the intervals.

5 An inequality of Rubio de Francia and applications

We start by recalling the following one-sided estimate due to Rubio de Francia [20].

Theorem 5.1 *[20] Let $2 \le p < \infty$. Let (I_j) be an arbitrary sequence of pairwise disjoint intervals in $\mathbb{Z}$. Then*

$$\left\| \left\| \left(\mathbb{P}_{I_j} f \right)_{j\in\mathbb{N}} \right\|_{l^2(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \le C(p) \|f\|_{L^p(\mathbb{T})}, \ \forall f \in \mathscr{P}(\mathbb{T}).$$

Here, $C(p)$ is independent of the choice of the disjoint intervals.

In the remaining part of this section, we establish two consequences of Theorem 5.1 (Corollaries 5.3 and 5.5). These results will play a crucial role in the proof of Theorem 1.1.

Lemma 5.2 *Let $2 \le q \le p < \infty$. Consider, in each dyadic set Δ_j^1, a family of $k(j)$ pairwise disjoint intervals $I_{j,r}$, $r \in [\![1, k(j)]\!]$. Then there exists $C(p,q)$ such that*

$$\left\| \left\| \left(\sum_r \left| \mathbb{P}_{I_{j,r}} f \right| \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \le C(p,q) \left\| \left\| \left((k(j))^{1/2} \mathbb{P}_{\Delta_j^1} f \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})}, \ \forall f \in \mathscr{P}(\mathbb{T}). \tag{5.1}$$

Though we will apply the above lemma with $k(j)$ as in Step 2 of the proof of Proposition 3.1 and to the equal length intervals considered there, we emphasize the fact that here $k(j)$ and the intervals are arbitrary.

Proof By the Cauchy-Schwarz inequality, we have, for all $j \in \mathbb{N}$,

$$\sum_r \left| \mathbb{P}_{I_{j,r}} f \right| \le (k(j))^{1/2} \left(\sum_r \left| \mathbb{P}_{I_{j,r}} f \right|^2 \right)^{1/2}. \tag{5.2}$$

In view of (5.2), (5.1) will follow from the estimate

$$\left\| \left\| \left(\left(\sum_r \left| \mathbb{P}_{I_{j,r}} f \right|^2 \right)^{1/2} \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \lesssim \left\| \left\| \left(\mathbb{P}_{\Delta_j^1} f \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \tag{5.3}$$

applied to the function $\sum (k(j))^{1/2} \mathbb{P}_{\Delta_j^1} f$.

In turn, estimate (5.3) will be obtained by interpolation, starting from the limiting cases $q = 2$ and $q = p$.

i. Proof of (5.3) *when* $q = 2$. When $q = 2$, (5.3) amounts to

$$\left\| \left\| \left(\mathbb{P}_{I_{j,r}} f \right)_{j\in\mathbb{N}, r \in [\![1,k(j)]\!]} \right\|_{l^2} \right\|_{L^p(\mathbb{T})} \lesssim \left\| \left\| \left(\mathbb{P}_{\Delta_j^1} f \right)_{j\in\mathbb{N}} \right\|_{l^2(\mathbb{N})} \right\|_{L^p(\mathbb{T})}. \tag{5.4}$$

By Theorem 5.1, the left hand side of (5.4) can be estimated by $\|f\|_{L^p(\mathbb{T})}$. By the square function theorem, the right hand side of (5.4) is equivalent to $\|f\|_{L^p(\mathbb{T})}$. This proves (5.3) for $q = 2$.

ii. Proof of (5.3) *when* $q = p$. In this case, (5.3) is equivalent to

$$\sum_{j\ge 0} \int_{\mathbb{T}} \left(\sum_r |\mathbb{P}_{I_{j,r}} f|^2 \right)^{p/2} \lesssim \sum_{j \ge 0} \int_{\mathbb{T}} |\mathbb{P}_{\Delta_j^1} f|^p.$$

The above estimate follows from

$$\int_{\mathbb{T}} \left(\sum_r |\mathbb{P}_{I_{j,r}} f|^2 \right)^{p/2} \lesssim \|\mathbb{P}_{\Delta_j^1} f\|^p_{L^p(\mathbb{T})}, \ \forall j \ge 0,$$

which in turn is a consequence of Theorem 5.1 applied to $\mathbb{P}_{\Delta_j^1} f$.

iii. Proof of (5.3) *when* $2 < q < p$. Let us consider the Banach space

$$X_{p,q} := \left\{ (f_j)_{j\in\mathbb{N}} \in L^p(\mathbb{T}, l^q(\mathbb{N})); \operatorname{supp} \widehat{f_j} \subset \Delta_j^1, \forall j \in \mathbb{N} \right\}.$$

Define the linear map

$$X_{p,q} \ni (f_j)_{j\in\mathbb{N}} \xrightarrow{T} \left((u_{j,r})_{r \in [\![1,k(j)]\!]} \right)_{j\in\mathbb{N}},$$

where

$$u_{j,r} = \begin{cases} \mathbb{P}_{I_{j,r}} f_j, & \text{if } r \in [\![1,k(j)]\!] \\ 0, & \text{otherwise} \end{cases}.$$

Up to now, we have proved (5.3) for $q = 2$ and $q = p$. Equivalently, we have proved that T is continuous from $X_{p,2}$ into $L^p(\mathbb{T}, l^2(\mathbb{N}, l^2(\mathbb{N})))$ and from $X_{p,p}$ into $L^p(\mathbb{T}, l^p(\mathbb{N}, l^2(\mathbb{N})))$, with norms controlled by a constant.

Assume for a while that, if q and θ satisfy $\dfrac{1}{q} = \dfrac{1-\theta}{2} + \dfrac{\theta}{p}$, then

$$X_{p,q} \backsimeq (X_{p,2}, X_{p,p})_\theta. \tag{5.5}$$

We continue as follows: we have $(l^2(\mathbb{N},l^2(\mathbb{N})),l^p(\mathbb{N},l^2(\mathbb{N})))_\theta = l^q(\mathbb{N},l^2(\mathbb{N}))$, which implies

$$(L^p(\mathbb{T},l^2(\mathbb{N},l^2(\mathbb{N}))),L^p(\mathbb{T},l^p(\mathbb{N},l^2(\mathbb{N}))))_\theta = L^p(\mathbb{T},l^q(\mathbb{N},l^2(\mathbb{N}))).$$

Note that the interpolation results used in this part of the proof can be found in [2, Theorem 5.1.2]. By complex interpolation, T is continuous from $(X_{p,2},X_{p,p})_\theta$ into $L^p(\mathbb{T},l^q(\mathbb{N},l^2(\mathbb{N})))$. In view of (5.5), T maps continuously $X_{p,q}$ into $L^p(\mathbb{T},l^q(\mathbb{N},l^2(\mathbb{N})))$. This is precisely (4.8) and the conclusion of Lemma 5.2.

Let us now prove (5.5). By Theorem 4.9, the projection map

$$L^p(\mathbb{T},l^q(\mathbb{N})) \ni (f_j)_{j\in\mathbb{N}} \xrightarrow{\pi} \left(\mathbb{P}_{\Delta^1_j} f_j\right)_{j\in\mathbb{N}} \in X_{p,q}$$

is well-defined and continuous. Clearly, π is onto and the continuous embedding $J : X_{p,q} \to L^p(l^q)$ is a right inverse of π. By complex interpolation, π is continuous from the space $(L^p(\mathbb{T},l^2(\mathbb{N})),L^p(\mathbb{T},l^p(\mathbb{N})))_\theta$ into the space $(X_{p,2},X_{p,p})_\theta$, and the continuous embedding $J : (X_{p,2},X_{p,p})_\theta \to (L^p(\mathbb{T},l^2(\mathbb{N})),L^p(\mathbb{T},l^p(\mathbb{N})))_\theta$ is a right inverse of π. Using the equality

$$(L^p(\mathbb{T},l^2(\mathbb{N})),L^p(\mathbb{T},l^p(\mathbb{N})))_\theta = L^p(\mathbb{T},l^q(\mathbb{N})),$$

we find that (5.5) holds. □

The Fubini theorem combined with Lemma 5.2 leads to the following

Corollary 5.3 *Let $2 \le q \le p < \infty$. Consider, in each dyadic set Δ^1_j, a family of $k(j)$ pairwise disjoint intervals $I_{j,r}$, $r \in [\![1,k(j)]\!]$. Then there exists $C(p,q)$ such that, for each $f \in \mathscr{P}(\mathbb{T}^d)$, we have*

$$\left\|\left\|\left(\sum_r \left|\mathbb{P}_{I_{j,r}\times\mathbb{Z}^{d-1}} f\right|\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)} \le C(p,q)\left\|\left\|\left((k(j))^{1/2}\mathbb{P}_{\Delta^1_j\times\mathbb{Z}^{d-1}} f\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T}^d)}. \tag{5.6}$$

In contrast with the hypotheses of Lemma 5.2, in the next result the intervals $I_{j,r}$ are not arbitrary.

Lemma 5.4 *Assume that $2 \le q \le p < \infty$. Consider, in each dyadic set Δ^1_j, a family of $k(j)$ pairwise disjoint intervals $I_{j,r}$, $r \in [\![1,k(j)]\!]$, all of same cardinal $l(j) = 2^{m(j)}$, with $m(j) \in \mathbb{N}$. Then there exists $C(p,q)$ such that*

$$\left\|\left\|\left(\sum_r \left|\mathbb{P}_{I_{j,r}} f\right|'\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \le C(p,q)\left\|\left\|\left((k(j))^{1/2} l(j)\mathbb{P}_{\Delta^1_j} f\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})}, \ \forall f \in \mathscr{P}(\mathbb{T}). \tag{5.7}$$

Proof Arguing as at the beginning of the proof of Lemma 5.2, it suffices to establish the estimate

$$\left\|\left\|\left(\left(\sum_r \left(\left|\mathbb{P}_{I_{j,r}} f\right|'\right)^2\right)^{1/2}\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \lesssim \left\|\left\|\left(l(j)\mathbb{P}_{\Delta^1_j} f\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})}, \ \forall f \in \mathscr{P}(\mathbb{T}). \tag{5.8}$$

We write $I_{j,r} = [\![a_{j,r}, b_{j,r}-1]\!]$, where $b_{j,r} - a_{j,r} = l(j) = 2^{m(j)}$. We rely on the key inequality

$$\left|\left|\mathbb{P}_{I_{j,r}} f\right|'\right| \le l(j)\left|\mathbb{F}_{l(j)} * (e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f)\right|, \tag{5.9}$$

whose proof is postponed to the appendix; see Lemma 8.6. Taking (5.9) for granted, we obtain (5.8) provided the following inequality holds:

$$\left\|\left\|\left(\left(\sum_r \left|\mathbb{F}_{l(j)} * (e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f)\right|^2\right)^{1/2}\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \lesssim \left\|\left\|\left(\mathbb{P}_{\Delta^1_j} f\right)_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})}\right\|_{L^p(\mathbb{T})}. \tag{5.10}$$

Indeed, (5.8) follows by applying (5.10) to $\sum l(j)\mathbb{P}_{\Delta^1_j} f$ and using (5.9).

We now turn to (5.10). It suffices to establish the validity of this estimate in the limiting cases $q = 2$ and $q = p$. Indeed, if this is proved then we are in position to repeat the interpolation argument used in the proof of Lemma 5.2.[8]

i. Proof of (5.10) *when* $q = 2$. We have to prove the inequality

$$\left\|\left\|\left(\mathbb{F}_{l(j)} * (e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f)\right)_{j\in\mathbb{N}, r\in[\![1,k(j)]\!]}\right\|_{l^2(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \lesssim \left\|\left\|\left(\mathbb{P}_{\Delta^1_j} f\right)_{j\in\mathbb{N}}\right\|_{l^2(\mathbb{N})}\right\|_{L^p(\mathbb{T})}.$$

This is obtained as follows: we have

$$\left\|\left\|\left(\mathbb{F}_{l(j)} * (e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f)\right)_{j\in\mathbb{N}, r\in[\![1,k(j)]\!]}\right\|_{l^2(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \overset{(a)}{\lesssim} \left\|\left\|\left(e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f\right)_{j\in\mathbb{N}, r\in[\![1,k(j)]\!]}\right\|_{l^2(\mathbb{N})}\right\|_{L^p(\mathbb{T})} \overset{(b)}{\lesssim} \left\|\left\|\left(\mathbb{P}_{\Delta^1_j} f\right)_{j\in\mathbb{N}}\right\|_{l^2(\mathbb{N})}\right\|_{L^p(\mathbb{T})}.$$

Here,

(a) is a consequence of Lemma 4.3.
(b) follows from Theorem 5.1 and the square function theorem (2.2).

[8] Interpolation is applied to the operator $f \mapsto \left(\left(\mathbb{F}_{l(j)} * (e^{-\imath a_{j,r}x}\mathbb{P}_{I_{j,r}} f)\right)_{r\in[\![1,k(j)]\!]}\right)_{j\in\mathbb{N}}$.

ii. Proof of (5.10) *when* $q = p$. In this case, the left-hand side of (5.10), denoted by A, is estimated as follows

$$A = \left\| \left(\left\| \left(\sum_r \left| \mathbb{F}_{l(j)} * (e^{-\imath a_{j,r} x} \mathbb{P}_{I_{j,r}} f) \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \right)_{j\in\mathbb{N}} \right\|_{l^p(\mathbb{N})} \overset{(a)}{\lesssim} \left\| \left(\left\| \left(\sum_r \left| e^{-\imath a_{j,r} x} \mathbb{P}_{I_{j,r}} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \right)_{j\in\mathbb{N}} \right\|_{l^p(\mathbb{N})}$$

$$= \left\| \left(\left\| \left(\sum_r \left| \mathbb{P}_{I_{j,r}} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \right)_{j\in\mathbb{N}} \right\|_{l^p(\mathbb{N})} \overset{(b)}{\lesssim} \left\| \left\| \left(\mathbb{P}_{\Delta_j^1} f \right)_{j\in\mathbb{N}} \right\|_{L^p(\mathbb{T})} \right\|_{l^p(\mathbb{N})} = \left\| \left\| \left(\mathbb{P}_{\Delta_j^1} f \right)_{j\in\mathbb{N}} \right\|_{l^p(\mathbb{N})} \right\|_{L^p(\mathbb{T})};$$

items (a) and (b) are justified as above. □

Corollary 5.5 *Assume that* $2 \le q \le p < \infty$. *Consider, in each dyadic set* Δ_j^1, *a family of* $k(j)$ *pairwise disjoint intervals* $I_{j,r}$, $r \in [\![1, k(j)]\!]$, *all of same cardinal* $l(j) = 2^{m(j)}$, *with* $m(j) \in \mathbb{N}$. *Then there exists* $C(p,q)$ *such that, for each* $f \in \mathscr{P}(\mathbb{T}^d)$, *we have*

$$\left\| \left\| \left(\sum_r \partial_1 \left| \mathbb{P}_{I_{j,r} \times \mathbb{Z}^{d-1}} f \right| \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \le C(p,q) \left\| \left\| \left((k(j))^{1/2} l(j) \mathbb{P}_{\Delta_j^1 \times \mathbb{Z}^{d-1}} f \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)}. \tag{5.11}$$

Remark 5.6 The proof of Lemma 5.4 also yields the following more general result. Assume that $2 \le q \le p < \infty$. Consider, in each dyadic set Δ_j^1, a family of $k(j)$ pairwise disjoint intervals $I_{j,r}$, $r \in [\![1, k(j)]\!]$, each of cardinal $l(j,r) = 2^{m(j,r)}$, with $m(j,r) \in \mathbb{N}$. Then there exists $C(p,q)$ such that

$$\left\| \left\| \left(\sum_r \left| \mathbb{P}_{I_{j,r}} f \right|' \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})} \le C(p,q) \left\| \left\| \left((k(j))^{1/2} \sum_r l(j,r) \mathbb{P}_{I_{j,r}} f \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T})}, \quad \forall f \in \mathscr{P}(\mathbb{T}). \tag{5.12}$$

6 Proof of Proposition 3.1

Proof of Lemma 3.3.

i. Proof of (3.11). Recall that supp $\hat{f} \subset \mathbb{B}$. Since in $\mathbb{B}_j$ we have $2^{j-1} \le n_1 < 2^j$, we find that

$$\left(2^{(s+1)j} F_j \right)_{j\in\mathbb{N}} = T_\varphi \left(\left(2^{sj} f_j \right)_{j\in\mathbb{N}} \right). \tag{6.1}$$

Here, φ is the scalar function (acting on vector-valued functions) $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_d$, where $\varphi_1(n_1) = \dfrac{2^j}{\imath n_1}$ if $2^{j-1} \le n_1 < 2^j$, 0 otherwise and $\varphi_k \equiv 1$ for $k \in [\![2, d]\!]$. By combining (6.1) with Corollary 4.6, we find that

$$\|F\|_{F_q^{s+1,p}} = \left\| \left\| \left(2^{(s+1)j} F_j \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} = \left\| \left\| T_\varphi \left(2^{sj} f_j \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \lesssim \left\| \left\| \left(2^{sj} f_j \right)_{j\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} = \|f\|_{F_q^{s,p}}.$$

ii. Proof of (3.12). Since $\mathbb{P}_{\Delta_k^d}(F_j) = \delta_{jk} F_j$ and $\mathbb{P}_{\Delta_k^d}(f_j) = \delta_{jk} f_j$, we have

$$\|F_j\|_{F_q^{s,p}(\mathbb{T}^d)} = 2^{sj} \|F_j\|_{L^p(\mathbb{T}^d)} \quad \text{and} \quad \|f_j\|_{F_q^{s,p}(\mathbb{T}^d)} = 2^{sj} \|f_j\|_{L^p(\mathbb{T}^d)}. \tag{6.2}$$

Corollary 4.7 applied to $\varphi_1(n_1) = \dfrac{1}{\imath n_1}$ if $n_1 \ge 1$ and $\varphi_1(n_1) = 0$ otherwise and $\varphi_k \equiv 1$ for $k \in [\![2, d]\!]$ implies that

$$\|F_j\|_{L^p(\mathbb{T}^d)} \lesssim 2^{-j} \|f_j\|_{L^p(\mathbb{T}^d)}. \tag{6.3}$$

Lemma 8.2 in the appendix combined with (1.2) and (6.3) yields

$$\|F_j\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^{jd/p} \|F_j\|_{L^p(\mathbb{T}^d)} \lesssim 2^{j(d/p-1)} \|f_j\|_{L^p(\mathbb{T}^d)} = \|f_j\|_{F_q^{s,p}(\mathbb{T}^d)}. \tag{6.4}$$

iii. Proof of (3.13). Lemma 8.2 combined with (6.4) gives

$$\|\nabla F_j\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^j \|F_j\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^j \|f_j\|_{F_q^{s,p}(\mathbb{T}^d)}.$$

iv. Proof of (3.14). Since $f_j = \mathbb{P}_{\Delta_j^d} f$, we have

$$\|f\|^p_{F_q^{s,p}(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \left(\sum_k 2^{skq} |\mathbb{P}_{\Delta_k^d} f|^q \right)^{p/q} \ge 2^{sjp} \int_{\mathbb{T}^d} |\mathbb{P}_{\Delta_j^d} f|^p = 2^{sjp} \int_{\mathbb{T}^d} |f_j|^p = \|f_j\|^p_{F_q^{s,p}(\mathbb{T}^d)}. \square$$

Proof of Lemma 3.4.

i. Proof of (3.23). Using the fact that $\mathbb{B}_{j,r} = \left(\llbracket a_{j,r}, b_{j,r}-1\rrbracket \times \mathbb{Z}^{d-1}\right) \cap \mathbb{B}_j$ for all $1 \le r \le k(j)$, we obtain that

$$I := \sum_{1\le r\le k(j)} \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|^p_{L^p(\mathbb{T}^d)}$$

satisfies

$$\begin{aligned} I &= \sum_{1\le r\le k(j)} \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} \left| \sum_{n_1\in\llbracket a_{j,r}, b_{j,r}-1\rrbracket} \widehat{(f_j)_{*x'}}(n_1) e^{\imath n_1 x_1} \right|^p dx_1\, dx' \\ &\overset{(a)}{\le} \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} \left(\sum_{1\le r\le k(j)} \left| \sum_{n_1\in\llbracket a_{j,r}, b_{j,r}-1\rrbracket} \widehat{(f_j)_{*x'}}(n_1) e^{\imath n_1 x_1} \right|^2 \right)^{p/2} dx_1\, dx' \overset{(b)}{\lesssim} \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} |(f_j)_{*x'}|^p\, dx_1\, dx' = \|f_j\|^p_{L^p(\mathbb{T}^d)}. \end{aligned}$$

Here,

(a) is a consequence of $\sum |a_r|^p \le \left(\sum |a_r|^2\right)^{p/2}$, valid when $p \ge 2$.
(b) follows from Theorem 5.1 applied to $I_r^1 = \llbracket a_{j,r}, b_{j,r}-1\rrbracket$ and $f = (f_j)_{*x'}$.

This completes the proof of (3.23).

ii. Proof of (3.24). When $j \le l$, this is just (3.12). If $j > l$, by Hölder's inequality we have

$$\|G_j\|_{L^\infty(\mathbb{T}^d)} \le k(j)^{1/p'} \left(\sum_{1\le r\le k(j)} \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \frac{1}{n_1} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|^p_{L^\infty(\mathbb{T}^d)} \right)^{1/p}. \tag{6.5}$$

Corollary 4.7 applied to $\varphi_1(n_1) = \dfrac{1}{n_1}$ when $n_1 \ge 1$, 0 otherwise, and $\varphi_j \equiv 1$ for $j \in \llbracket 2, d\rrbracket$ implies that

$$\left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \frac{1}{n_1} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|_{L^p(\mathbb{T}^d)} \lesssim 2^{-j} \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|_{L^p(\mathbb{T}^d)}. \tag{6.6}$$

Corollary 8.3 applied to $x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \frac{1}{n_1} \widehat{f}_j(n) e^{\imath n\cdot x}$ and combined with (1.2), (6.2), (6.5), (6.6) and (3.23) yields

$$\begin{aligned} \|G_j\|_{L^\infty(\mathbb{T}^d)} &\lesssim k(j)^{1/p'} \left(\sum_{1\le r\le k(j)} \left(2^{j(d-1)/p} (l(j))^{1/p} \right)^p \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \frac{1}{n_1} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|^p_{L^p(\mathbb{T}^d)} \right)^{1/p} \\ &\lesssim k(j)^{1/p'} l(j)^{1/p} 2^{j(d-1)/p} 2^{-j} \left(\sum_{1\le r\le k(j)} \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|^p_{L^p(\mathbb{T}^d)} \right)^{1/p} \\ &\lesssim k(j)^{1-2/p} 2^{sj} \left(\sum_{1\le r\le k(j)} \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|^p_{L^p(\mathbb{T}^d)} \right)^{1/p} \lesssim k(j)^{1-2/p} 2^{sj} \|f_j\|_{L^p(\mathbb{T}^d)} = k(j)^{1-2/p} \|f_j\|_{F^{s,p}_q(\mathbb{T}^d)}. \end{aligned}$$

In the third inequality, we used the fact that $k(j)l(j) = 2^{j-1}$.

iii. Proof of (3.25). The proof is similar to the one of (3.24). When $j \le l$, (3.25) is a consequence of (3.13). When $j > l$, we start from the inequality

$$|\partial_k G_j(x)| \le \sum_{1\le r\le k(j)} \left| \sum_{n\in\mathbb{B}_{j,r}} \frac{n_k}{n_1} \widehat{f}(n) e^{\imath n\cdot x} \right|.$$

We continue as in the proof of (3.24). The only noticeable difference is that, in the proof, estimate (6.6) is replaced with

$$\left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \frac{n_k}{n_1} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|_{L^p(\mathbb{T}^d)} \lesssim \left\| x \mapsto \sum_{n\in\mathbb{B}_{j,r}} \widehat{f}_j(n) e^{\imath n\cdot x} \right\|_{L^p(\mathbb{T}^d)}. \tag{6.7}$$

In turn, (6.7) with $k \ne 1$ follows from Corollary 4.7 applied to $\varphi_1(n_1) = \dfrac{1}{n_1}$, $\varphi_k(n_k) = n_k$, $\varphi_m(n_m) \equiv 1$ for $m \ne 1, k$. □

Proof of Lemma 3.5.

i. Proof of (3.28) *and* (3.29). Clearly, (3.29) is a consequence of (3.28). By straightforward induction on $m \ge 0$, we find that

$$\sum_{j=0}^m F_j \prod_{j<k\le m} (1 - H_k) = \sum_{j=0}^m F_j - \sum_{j=0}^m H_j K_j. \tag{6.8}$$

Noting that the sum in (3.28) contains only a finite number of nonzero terms, we find that (3.28) is a consequence of (6.8).

ii. Proof of (3.30)*,* (3.31) *and* (3.32)*.* Inequality (3.32) follows immediately from (3.20) and (3.26) provided η_0 is sufficiently small.
In order to establish (3.30), we start from (3.20), which implies that $|F_j| \le H_j$. This yields

$$|Y_1| \le \sum_j H_j \prod_{k>j} (1-H_k), \tag{6.9}$$

with $0 \le H_k \le 1$. Estimate (3.30) is a consequence of (6.9) and of the inequality

$$\sum_j a_j \prod_{k>j}(1-a_k) \le 1, \text{ valid if } a_j \in [0,1],\ \forall j \ge 0.$$

The proof of (3.31) is similar to the one of (3.30). □

Proof of Lemma 3.6. In what follows, we use the convention $H_m = K_m = 0$ when $m < 0$.
In view of (3.19), we have supp $\widehat{K_j} \subset [|n| \le 2^j - 2]$. Using again (3.19), we have supp $\widehat{H_j K_j} \subset [|n| \le 2^{j+1} - 3]$. Thus

$$\sum_{j\ge 0} \partial_1 (H_j K_j) = \sum_{j\ge 0} \sum_{0\le m\le j+1} \mathbb{P}_{\Delta_m^d} (\partial_1 (H_j K_j)) = \sum_{m\ge 0} \sum_{j \ge 0} \mathbb{P}_{\Delta_m^d} (\partial_1 (H_{m+j-1} K_{m+j-1})).$$

Hence,

$$\left\| \sum_{j\ge 0} \partial_1 (H_j K_j) \right\|_{F_q^{s,p}(\mathbb{T}^d)} \le \sum_{j\ge 0} \left\| \sum_{m\ge 0} \mathbb{P}_{\Delta_m^d} (\partial_1 (H_{m+j-1} K_{m+j-1})) \right\|_{F_q^{s,p}(\mathbb{T}^d)} \equiv \sum_{j \ge 0} S_j. \tag{6.10}$$

Note that

$$S_j = \left\| \left\| \left(2^{sm} \mathbb{P}_{\Delta_m^d} (\partial_1 (H_{m+j-1} K_{m+j-1})) \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)}. \tag{6.11}$$

We are going to estimate the sum in (6.10) by an interpolation technique: we will establish two estimates involving S_j (estimates (6.13) and (6.20) below) and use the most convenient one according to the values of j.
These two estimates will rely on:

$$\left\| \left\| \left(2^{(s+1)m} H_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \lesssim \varepsilon^{-1/2} \|f\|_{F_q^{s,p}}. \tag{6.12}$$

Let us prove (6.12):

$$\begin{aligned}
\left\| \left\| \left(2^{(s+1)m} H_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} &\overset{(a)}{\lesssim} \left\| \left\| \left(2^{(s+1)m} G_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \overset{(b)}{\lesssim} \left\| \left\| \left(2^{(s+1)m} k(m)^{1/2} F_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \\
&\le \varepsilon^{-1/2} \left\| \left\| \left(2^{(s+1)m} F_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \le \varepsilon^{-1/2} \|F\|_{F_q^{s+1,p}} \overset{(c)}{\lesssim} \varepsilon^{-1/2} \|f\|_{F_q^{s,p}}.
\end{aligned}$$

The above estimates are obtained as follows:

(a) is obtained by combining Lemma 4.4 with (3.16) and (3.22).
(b) is a consequence of Corollary 5.3 applied to the function $\sum_m 2^{(s+1)m} F_{m+j-1}$.
(c) relies on (3.11).

This ends the proof of (6.12).

i. First estimate of S_j. Starting from (6.11), we obtain successively

$$\begin{aligned}
S_j &\overset{(a)}{\lesssim} \left\| \left\| \left(2^{(s+1)m} \mathbb{P}_{\Delta_m^d} (H_{m+j-1} K_{m+j-1}) \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \overset{(b)}{\lesssim} \left\| \left\| \left(2^{(s+1)m} H_{m+j-1} K_{m+j-1} \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \\
&\overset{(c)}{\le} \left\| \left\| \left(2^{(s+1)m} H_{m+j-1} \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \overset{(d)}{\lesssim} \varepsilon^{-1/2} 2^{-(s+1)j} \|f\|_{F_q^{s,p}}.
\end{aligned}$$

The above estimates are obtained as follows:

(a) follows from Lemma 4.8.
(b) is a consequence of Theorem 4.9.
(c) relies on (3.31).
(d) relies on (6.12).

We established our first estimate:

$$S_j \lesssim \varepsilon^{-1/2} 2^{-(s+1)j} \|f\|_{F_q^{s,p}}. \tag{6.13}$$

ii. Second estimate of S_j. Using (6.11) and Theorem 4.9, we find that

$$\begin{aligned}
S_j &\lesssim \left\| \left\| (2^{sm} \partial_1 (H_{m+j-1} K_{m+j-1}))_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \\
&\le \left\| \left\| (2^{sm} K_{m+j-1} \partial_1 H_{m+j-1})_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} + \left\| \left\| (2^{sm} H_{m+j-1} \partial_1 K_{m+j-1})_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \\
&\lesssim 2^{-sj} \left\| \left\| (2^{sm} K_m \partial_1 H_m)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} + 2^{-sj} \left\| \left\| (2^{sm} H_m \partial_1 K_m)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \equiv 2^{-sj} (T_j + U_j).
\end{aligned} \tag{6.14}$$

We proceed by estimating T_j and U_j separately.

iii. Estimate of T_j. Let $l \in \mathbb{N}$ be such that $\varepsilon = 2^{-l}$. Using (3.21) and the definition of K_j as well as (3.31) and (3.18), we find

$$\begin{aligned} T_j &\le \left\| \left\| (2^{sm} K_m \partial_1 H_m)_{m\le l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)} + \left\| \left\| (2^{sm} K_m \partial_1 H_m)_{m> l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)} \\ &\le \left\| \left\| \left(2^{sm} \left| \sum_{k=1}^{m} G_k \right| \partial_1 H_m \right)_{m\le l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)} + \left\| \left\| (2^{sm} \partial_1 H_m)_{m> l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)} \\ &\overset{(a)}{\lesssim} l \|f\|_{F^{s,p}_q(\mathbb{T}^d)} \left\| \left\| (2^{sm} \partial_1 H_m)_{m\le l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)} + \left\| \left\| (2^{sm} \partial_1 H_m)_{m> l} \right\|_{l^q} \right\|_{L^p(\mathbb{T}^d)}. \end{aligned} \tag{6.15}$$

Inequality (a) follows from the fact that, when $m \le l$, we have

$$\left| \sum_{k=1}^{m-1} G_k \right| \lesssim \sum_{k=1}^{m-1} \|f_k\|_{F^{s,p}_q(\mathbb{T}^d)} \lesssim l \|f\|_{F^{s,p}_q(\mathbb{T}^d)}; \tag{6.16}$$

here, we rely on (3.24) and (3.14).

For $I = [\![0,l]\!]$ or $I = [\![l+1,\infty[\![$, we have

$$\left\| \left\| (2^{sm} \partial_1 H_m)_{m\in I} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \overset{(a)}{\lesssim} \left\| \left\| (2^{sm} \partial_1 G_m)_{m\in I} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \overset{(b)}{\lesssim} \left\| \left\| \left(2^{sm} k(m)^{1/2} l(m) F_m \right)_{m\in I} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)}. \tag{6.17}$$

The above estimates are obtained as follows:

(a) follows from Lemma 4.4.
(b) is a consequence of Corollary 5.5 applied to $\sum_m 2^{sm} F_m$.

Let us recall that, when $I = [\![0,l]\!]$, we have $k(m) = 1$ and $l(m) = 2^{m-1}$ for any $m \in I$. On the other hand, in the case where $I = [\![l+1,\infty[\![$, we have $k(m) = \varepsilon^{-1} = 2^l$ and $l(m) = \varepsilon 2^{m-1} = 2^{m-l-1}$. By combining this with (6.15), (6.17), (3.11) and with the definition of F_m, we obtain

$$T_j \lesssim \left(-\ln\varepsilon \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{1/2} \right) \|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{6.18}$$

iv. Estimate of U_j. We start by estimating

$$\begin{aligned} |\partial_1 K_m| &\overset{(a)}{\lesssim} \sum_{n<m} \left(\|\nabla F_n\|_{L^\infty(\mathbb{T}^d)} + \|\nabla H_n\|_{L^\infty(\mathbb{T}^d)} \right) \overset{(b)}{\lesssim} \sum_{n<m} \left(2^{n(1+d/p)} \|F_n\|_{L^p(\mathbb{T}^d)} + 2^n k(n)^{1-2/p} \|f_n\|_{F^{s,p}_q(\mathbb{T}^d)} \right) \\ &\overset{(c)}{\lesssim} \sum_{n<m} k(n)^{1-2/p} 2^n \|f_n\|_{F^{s,p}_q(\mathbb{T}^d)} \overset{(d)}{\lesssim} \varepsilon^{2/p-1} 2^m \|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \end{aligned}$$

Here are the explanations:

(a) follows from the definition of K_m and (3.32).
(b) is a consequence of Lemma 8.2 and (3.27).
(c) relies on (6.2) and (6.3).
(d) is a consequence of the fact that $k(m) \le \varepsilon^{-1}$ for any $m \ge 1$.

In view of the definition of U_j, this implies that

$$U_j \lesssim \varepsilon^{2/p-1} \|f\|_{F^{s,p}_q(\mathbb{T}^d)} \left\| \left\| \left(2^{(s+1)m} H_m \right)_{m\in\mathbb{N}} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{T}^d)} \lesssim \varepsilon^{2/p-3/2} \|f\|^2_{F^{s,p}_q(\mathbb{T}^d)}, \tag{6.19}$$

where the second inequality follows from (6.12). By combining (6.18) and (6.19), we obtain the second estimate of S_j :

$$S_j \lesssim 2^{-sj} \left((-\ln\varepsilon + \varepsilon^{2/p-3/2}) \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{1/2} \right) \|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{6.20}$$

v. Optimization. In view of (6.10), we have for every $\tau \ge 1$

$$\left\| \sum_{j\ge 0} \partial_1 (H_j K_j) \right\|_{F^{s,p}_q(\mathbb{T}^d)} \le \sum_{j\ge\tau} S_j + \sum_{j<\tau} S_j. \tag{6.21}$$

In the first sum, we use the first estimate of S_j, while in the second sum we use the second estimate. This gives

$$\begin{aligned} \left\| \sum_{j\ge 0} \partial_1 (H_j K_j) \right\|_{F^{s,p}_q(\mathbb{T}^d)} &\lesssim \max(\tau, 2^{-s\tau}) \left((-\ln\varepsilon + \varepsilon^{2/p-3/2}) \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{1/2} \right) \times \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{-1/2} 2^{-(s+1)\tau} \|f\|_{F^{s,p}_q} \\ &\lesssim \max(\tau, 2^{-s\tau}) \left(\varepsilon^{2/p-3/2} \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{1/2} \right) \|f\|_{F^{s,p}_q(\mathbb{T}^d)} + \varepsilon^{-1/2} 2^{-(s+1)\tau} \|f\|_{F^{s,p}_q(\mathbb{T}^d)} \\ &= \delta \left[\max(\tau, 2^{-s\tau}) \left(\varepsilon^{2/p-3/2} \delta + \varepsilon^{1/2} \right) + \varepsilon^{-1/2} 2^{-(s+1)\tau} \right], \end{aligned} \tag{6.22}$$

with $\delta := \|f\|_{F_q^{s,p}(\mathbb{T}^d)}$. We now choose l (and thus ε): we let l be the integer part of $\dfrac{-p\ln\delta}{2(p+1)\ln 2}$. Hence $\dfrac{1}{2}\delta^{p/(2p-2)} < \varepsilon = 2^{-l} \le \delta^{p/(2p-2)}$.

The next step consists in choosing the integer τ. Assume first that $s \ge 0$. Then we define and τ as the integer part of $\dfrac{l}{s+1}$, and an elementary computation gives

$$\left\|\sum_{j\ge 0}\partial_1(H_jK_j)\right\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(\mathbb{T}^d)}^{1+p/(4p-4)}\left(1-\ln\|f\|_{F_q^{s,p}(\mathbb{T}^d)}\right).$$

Assume next that $-\dfrac{1}{2} < s < 0$. We then let $\tau = l$ and obtain

$$\left\|\sum_{j\ge 0}\partial_1(H_jK_j)\right\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(\mathbb{T}^d)}^{1+(2s+1)p/(4p-4)}.$$

Taking δ_0 sufficiently small, we can thus ensure that (3.33) holds for any

$$0 < \alpha < \min\left(\frac{p}{4p-4}, \frac{(2s+1)p}{4p-4}\right)$$

and any $\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \delta_0$.

Moreover, the condition $k(j)^{1-2/p}\,\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \eta_0$ is satisfied provided that

$$\varepsilon^{1-2/p}\,\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \eta_0. \tag{6.23}$$

In turn, (6.23) is satisfied for small δ_0; this follows from the inequality

$$\varepsilon^{1-2/p}\,\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \|f\|_{F_q^{s,p}(\mathbb{T}^d)}^{(3p-4)/(2p-2)} \le \delta_0^{(3p-4)/(2p-2)}.$$

This completes the proof of Lemma 3.6. □

Proof of Lemma 3.7. We have

$$\|\nabla \mathbb{Y}_1\|_{F_q^{s,p}(\mathbb{T}^d)} \le \left\|\sum_j \nabla F_j\right\|_{F_q^{s,p}(\mathbb{T}^d)} + \left\|\sum_j \nabla(H_jK_j)\right\|_{F_q^{s,p}(\mathbb{T}^d)} := A + B.$$

By Lemma 3.6, we have that

$$\left\|\sum_j \partial_1(H_jK_j)\right\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \frac{1}{2d}$$

provided that $\|f\|_{F_q^{s,p}} \le \delta_0$ and δ_0 is chosen sufficiently small. A proof similar to the one of (6.13) combined with the choice of ε leads to

$$\left\|\sum_j \partial_m(H_jK_j)\right\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \varepsilon^{-1/2}\,\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \delta^{(3p-4)/(4p-4)}$$

for $m \in [\![2,d]\!]$. Hence $B \le \dfrac{1}{2}$ for small δ_0.

In order to estimate A, we prove that

$$\left\|\sum_j \nabla F_j\right\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(\mathbb{T}^d)}. \tag{6.24}$$

This is obtained as follows: for $m \in [\![1,d]\!]$, we have

$$\begin{aligned}\left\|\sum_j \partial_m F_j\right\|_{F_q^{s,p}(\mathbb{T}^d)} &= \left\|\left(\sum_k 2^{ksq}\left|\mathbb{P}_{\Delta_k^d}\left(\sum_j \partial_m F_j\right)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{T}^d)} = \left\|\left(\sum_k 2^{ksq}|\partial_m F_k|^q\right)^{1/q}\right\|_{L^p(\mathbb{T}^d)}\\ &\lesssim \left\|\left(\sum_k 2^{ksq}2^{qk}|F_k|^q\right)^{1/q}\right\|_{L^p(\mathbb{T}^d)} = \|F\|_{F_q^{s+1,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(\mathbb{T}^d)},\end{aligned}$$

by Lemma 4.8. This implies (6.24).

Finally, we have $\|\nabla \mathbb{Y}_1\|_{F_q^{s,p}(\mathbb{T}^d)} \le 1$, provided $\|f\|_{F_q^{s,p}(\mathbb{T}^d)} \le \delta_0$ with δ_0 sufficiently small. □

The proof of Proposition 3.1 is complete. □

7 Inversion of the divergence in domains

Throughout this section, Ω is a smooth bounded domain in $\mathbb{R}^d$. With no loss of generality, we may also assume that Ω is connected.

For the definition of the space $F_q^{s,p}(\mathbb{R}^d)$ and its basic properties, we refer the reader to [21, Chapter 4]. Let us recall that $F_q^{s,p}(\Omega)$ is the space of restrictions to Ω of elements in $F_q^{s,p}(\mathbb{R}^d)$.[9]

[9] When $s > 0$, functions in $F_q^{s,p}(\mathbb{R}^d)$ are locally integrable, and we deal with restrictions of functions. When $s \le 0$, we deal with restrictions of distributions.

We next give a meaning to the condition $\int_\Omega f = 0$, when $f \in F^{s,p}_q(\Omega)$. When $s > 0$, any element $f \in F^{s,p}_q(\Omega)$ belongs to $L^p(\Omega) \subset L^1(\Omega)$ so that the condition $\int_\Omega f = 0$ has an obvious meaning. When $s \le 0$, we have $-s < 1/p'$ (since $(s+1)p = d$). Therefore, we are in position to apply [21, Section 4.6.3, Theorem 1] and obtain that the characteristic function χ_Ω of Ω is a multiplier of $F^{-s,p'}_{q'}(\mathbb{R}^d)$. Equivalently, the extension operator by 0 of maps in $F^{-s,p'}_{q'}(\Omega)$, which will be denoted by $u \mapsto Eu$ in the sequel, is bounded from $F^{-s,p'}_{q'}(\Omega)$ into $F^{-s,p'}_{q'}(\mathbb{R}^d)$. In particular, there exists $C > 0$ such that for every $\varphi \in F^{-s,p'}_{q'}(\mathbb{R}^d)$ vanishing outside $\overline{\Omega}$ we have

$$\|\varphi\|_{F^{-s,p'}_{q'}(\mathbb{R}^d)} \le C\,\|\varphi_{|\Omega}\|_{F^{-s,p'}_{q'}(\Omega)}. \tag{7.1}$$

A consequence of this fact is that

$$F^{s,p}_q(\Omega) = (F^{-s,p'}_{q'}(\Omega))'. \tag{7.2}$$

Indeed, when $\Omega = \mathbb{R}^d$ this corresponds to [21, Section 2.1.5]. In order to prove (7.2) for a general Ω, let $f \in F^{s,p}_q(\Omega)$. For any extension $\widetilde{f} \in F^{s,p}_q(\mathbb{R}^d)$ of f and any $\varphi \in C^\infty_c(\Omega)$, define $\langle f, \varphi\rangle := \langle \widetilde{f}, E\varphi\rangle$. This definition does not depend on the choice of $\widetilde{f}$, and the linear functional thus defined is continuous on $C^\infty_c(\Omega)$ with respect to the $F^{-s',p'}_{q'}(\Omega)$ norm. The density of $C^\infty_c(\Omega)$ in $F^{-s,p'}_{q'}(\Omega)$ [21, Theorem 2.4.4.3] allows to extend this linear functional to a continuous linear functional on the whole $F^{-s,p'}_{q'}(\Omega)$ space. This proves the inclusion $F^{s,p}_q(\Omega) \subset (F^{-s,p'}_{q'}(\Omega))'$.
Conversely, let $g \in (F^{-s,p'}_{q'}(\Omega))'$. Thanks to the boundedness of E, $F^{-s,p'}_{q'}(\Omega)$ can be identified with a subspace of $F^{-s,p'}_{q'}(\mathbb{R}^d)$. The Hahn-Banach theorem yields an extension $\widetilde{g}$ of g as a bounded linear functional on $F^{-s,p'}_{q'}(\mathbb{R}^d)$ with the same norm as g. Thus $\widetilde{g} \in F^{s,p}_q(\mathbb{R}^d)$ [21, Proposition 2.1.5], which shows that $g \in F^{s,p}_q(\Omega)$.
The claim (7.2) is therefore proved. In addition, a closer look to the above arguments shows that the norms of $F^{s,p}_q(\Omega)$ and $(F^{-s,p'}_{q'}(\Omega))'$ are equivalent.

By the above discussion, we may interpret, when $s \le 0$, the condition $\int_\Omega f = 0$ as $\langle f, 1\rangle = 0$, where $\langle , \rangle$ denotes the pairing between $F^{s,p}_q(\Omega)$ and $F^{-s,p'}_{q'}(\Omega)$.

Proof of Theorem 1.3. The proof follows the one of [5, Theorem 2], with several modifications due to the possibly high regularity of functions in $F^{s,p}_q$ (whereas, in [5, Theorem 2], we only have $f \in L^p$). We first introduce the following notation: $Q'_r := (-r,r)^{d-1}$, $Q_r := Q'_r \times (0,r)$ $(r > 0)$ and $k := \max(0,[s])$. Here, $[\,]$ denotes the integer part. When we deal with traces, we identify Q'_r with $Q'_r \times \{0\}$.

i. Extension by reflection. Let $f \in F^{s,p}_q(Q_1)$. We introduce the invertible Vandermonde matrix $A := ((-1/j)^{i-1})_{1\le i,j\le k+1}$ and let

$$\alpha = {}^t(\alpha_1,\dots,\alpha_{k+1}) := A^{-1}[{}^t(1,\dots,1)] \in \mathbb{R}^{k+1}. \tag{7.3}$$

If $s > 0$, we then define, for $x = (x', x_d) \in Q'_1 \times (-1,1)$:

$$\tilde{f}(x',x_d) := \begin{cases} f(x',x_d), & \text{if } x_d > 0 \\ \sum_{i=1}^{k+1} \alpha_i f\left(x', -\frac{x_d}{i}\right), & \text{if } x_d < 0 \end{cases}.$$

Then $\tilde{f} \in F^{s,p}_q(Q'_1 \times (-1,1))$ and $\|\tilde{f}\|_{F^{s,p}_q(Q'_1\times(-1,1))} \lesssim \|f\|_{F^{s,p}_q(Q_1)}$. Indeed, this is easily checked when $s \in \mathbb{N}$ and $q = 2$ (that is, when $F^{s,p}_q$ is the classical Sobolev space $W^{s,p}$). This special case combined with interpolation implies that these assertions still hold when $s > 0$, $p > 1$ and $q > 1$.
When $s \le 0$, define $\tilde{f}$ in the following way:

$$\langle \tilde{f}, \varphi\rangle = \langle f, \varphi^e\rangle,\ \forall \varphi \in F^{-s,p'}_{q'}(Q'_1 \times (-1,1)),$$

where $\varphi^e(x',x_d) = \varphi(x',x_d) + \varphi(x',-x_d)$. Since the map

$$^e : F^{-s,p'}_{q'}(Q'_1 \times (-1,1)) \to F^{-s,p'}_{q'}(Q_1),\ \varphi \mapsto \varphi^e,$$

is continuous, it follows that the map

$$\tilde{} : F^{s,p}_q(Q_1) \to F^{s,p}_q(Q'_1 \times (-1,1)),\ f \mapsto \tilde{f}$$

is also continuous.

ii. Inversion of the divergence in the case of a flat boundary. This is performed in Lemma 7.2.

Lemma 7.1 *Let $f \in F^{s,p}_q(Q_1)$. Then there exists $\mathbb{X} \in F^{s+1,p}_q(Q_2) \cap L^\infty(Q_2)$ such that div $\mathbb{X} = f$ in Q_1 and tr $\mathbb{X}_d = 0$ on Q'_1. Moreover, there exists $C > 0$ (independent of f) such that*

$$\|\mathbb{X}\|_{F^{s+1,p}_q(Q_2)} + \|\mathbb{X}\|_{L^\infty(Q_2)} \le C\,\|f\|_{F^{s,p}_q(Q_1)}.$$

Proof Let g be a compactly supported $F_q^{s,p}$ extension of $\tilde{f}$ to $(-\pi,\pi)^d$. Extending g by periodicity, we may identify g with an element of $F_q^{s,p}(\mathbb{T}^d)$. We may further assume that $\int_{\mathbb{T}^d} g = 0$ and $\|g\|_{F_q^{s,p}(\mathbb{T}^d)} \lesssim \|f\|_{F_q^{s,p}(Q_1)}$. By Theorem 1.1, there exists $\mathbb{Y} \in F_q^{s+1,p}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ such that div $\mathbb{Y} = g$ and

$$\|\mathbb{Y}\|_{F_q^{s+1,p}(\mathbb{T}^d)} + \|\mathbb{Y}\|_{L^\infty(\mathbb{T}^d)} \le C\,\|g\|_{F_q^{s,p}(\mathbb{T}^d)}.$$

Let

$$B := \begin{pmatrix} 1 & 2 & \cdots & k+1 & -1 \\ 1 & & & & \alpha_1 \\ & 1 & & 0 & \alpha_2 \\ & & \ddots & & \vdots \\ & & & & \vdots \\ & 0 & & & \vdots \\ & & & 1 & \alpha_{k+1} \end{pmatrix},$$

where $\alpha_1,\dots,\alpha_{k+1}$ are given by (7.3). In order to prove that B is non singular, we note that

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} B = \begin{pmatrix} 1 & 2 & \cdots & k+1 & -1 \\ & & & & 1 \\ & & A & & \vdots \\ & & & & 1 \end{pmatrix}. \tag{7.4}$$

Therefore, it suffices to prove that the matrix in the right hand side of (7.4) is non singular. Its determinant equals

$$\Lambda = -\begin{vmatrix} p_1^{-1} & p_2^{-1} & \cdots & p_{k+1}^{-1} & 1 \\ p_1^0 & p_2^0 & \cdots & p_{k+1}^0 & 1 \\ \vdots & & & \vdots & \\ p_1^k & p_2^k & \cdots & p_{k+1}^k & 1 \end{vmatrix} = -\frac{1}{p_1\cdots p_{k+1}} V(p_1,\dots,p_{k+1},1),$$

where $p_j = -1/j$, $1 \le j \le k+1$ and $V(p_1,\dots,p_{k+1},1)$ is the Vandermonde determinant associated with the parameters $\{p_1,\dots,p_{k+1},1\}$. This implies that $\Lambda \ne 0$ and B is non singular.
We can thus define ${}^t(\beta_j)_{1\le j\le k+2} := B^{-1}[{}^t(0,1,0,\dots,0)]$. We then consider for any $1 \le i \le d-1$,

$$\mathbb{X}_i(x_1,\dots,x_d) := \sum_{j=1}^{k+1} \beta_j \mathbb{Y}_i\left(x', \frac{x_d}{j}\right) + \beta_{k+2}\mathbb{Y}_i(x',-x_d),$$

and for $i = d$,

$$\mathbb{X}_d(x_1,\dots,x_d) := \sum_{j=1}^{k+1} j\beta_j \mathbb{Y}_d\left(x', \frac{x_d}{j}\right) - \beta_{k+2}\mathbb{Y}_d(x',-x_d).$$

It is easy to check that div $\mathbb{X} = f$ in Q_1, that tr $X_d = 0$ on Q_1' and that $\|\mathbb{X}\|_{F_q^{s+1,p}(Q_2)} + \|\mathbb{X}\|_{L^\infty(Q_2)} \lesssim \|f\|_{F_q^{s,p}(Q_1)}$. □

Lemma 7.2 *Let $f \in F_q^{s,p}(Q_1)$. Then there exists $\mathbb{X} \in F_q^{s+1,p}(Q_2) \cap L^\infty(Q_2)$ such that div $\mathbb{X} = f$ in Q_1 and tr $\mathbb{X} = 0$ on Q_1'. Moreover, there exists $C > 0$ (independent of f) such that*

$$\|\mathbb{X}\|_{F_q^{s+1,p}(Q_2)} + \|\mathbb{X}\|_{L^\infty(Q_2)} \le C\,\|f\|_{F_q^{s,p}(Q_1)}.$$

Proof By Lemma 7.1, there exists $\mathbb{Y} \in F_q^{s+1,p}(Q_4) \cap L^\infty(Q_4)$ such that div $\mathbb{Y} = f$ on Q_1 and tr $\mathbb{Y}_d = 0$ on Q_1' (with the corresponding estimates). For $1 \le j \le d$, we set $\mathbb{C}_j := (-1)^{j+d}\mathbb{Y}_j$. Observe that tr $\mathbb{C}_d = 0$ on Q_1'. We claim that there exists $\mathbb{D}_j \in F_q^{s+2,p}(Q_2) \cap W^{1,\infty}(Q_2)$ such that tr $\mathbb{D}_j = 0$ on Q_2' and

$$\text{tr}\left(\frac{\partial \mathbb{D}_j}{\partial t}\right) = \mathbb{C}_j \text{ on } Q_2'.$$

Indeed, let $\mathbb{E}_j := \text{tr}_{|Q_4'}\mathbb{C}_j \in F_p^{s+1-1/p,p}(Q_4') \cap L^\infty(Q_4')$. Fix some $\rho \in C_c^\infty(\mathbb{R}^{d-1})$ such that $\int_{\mathbb{R}^{d-1}} \rho = 1$ and supp $\rho \subset Q_1'$. With the notation $\rho_t(x') = \frac{1}{t^{d-1}}\rho\left(\frac{x'}{t}\right)$, set, for $x' \in Q_2'$,

$$\mathbb{D}_j(x',t) := \begin{cases} t\mathbb{E}_j * \rho_t(x'), & \text{if } 0 < t < 2 \\ 0, & \text{if } t = 0 \end{cases}.$$

We then have for $1 \le k \le d-1$

$$\frac{\partial \mathbb{D}_j}{\partial x_k'}(x',t) = \mathbb{E}_j * \left(\frac{\partial \rho}{\partial x_k'}\right)_t$$

and also

$$\frac{\partial \mathbb{D}_j}{\partial t}(x',t) = \mathbb{E}_j * (\rho+\eta)_t(x'),$$

where $\eta(y') := -(d-1)\rho(y') - \langle \nabla\rho(y'), y'\rangle$. Clearly, $\mathbb{D}_j \in W^{1,\infty}(Q_2)$ and $\mathbb{D}_j(\cdot,0) = 0$. We now use the following result:

Lemma 7.3 *Let $L > 0$ and $R = \mathbb{R}^{d-1} \times (0,L)$. Let $\rho \in C_c^\infty(\mathbb{R}^{d-1})$ and $\sigma > 1/p$. For any $\gamma \in F_p^{\sigma-1/p,p}(\mathbb{R}^{d-1})$, let*

$$f_\gamma : R \to \mathbb{C},\ R \ni (x',t) \mapsto \gamma * \rho_t(x') \in \mathbb{C}.$$

Then the map $F_p^{\sigma-1/p,p}(\mathbb{R}^{d-1}) \ni \gamma \mapsto f_\gamma \in F_q^{\sigma,p}(R)$ is continuous for any $q \in [2,p]$.

Proof of Lemma 7.3. Let $K(x',x_d) := \rho_{x_d}(x')$. A straightforward computation shows that $\widehat{K}(\xi',x_d) = \widehat{\rho}(x_d\xi')$, where $\widehat{K}$ denotes the $d-1$ dimensional Fourier transform with respect to $x' = (x_1,...,x_{d-1})$. Since $\widehat{\rho} \in \mathscr{S}(\mathbb{R}^{d-1})$, the kernel $\widehat{K}$ belongs to the class S_0 introduced in [21, Section 3.2.1, Definition 2]. The conclusion of Lemma 7.3 follows therefore from [21, Theorem 3.2.2.3]. □

Proof of Lemma 7.2 continued. By Lemma 7.3 we have $\dfrac{\partial \mathbb{D}_j}{\partial x'_k}, \dfrac{\partial \mathbb{D}_j}{\partial t} \in F_q^{s+1,p}(Q_2)$, so that $\mathbb{D}_j \in F_q^{s+2,p}(Q_2)$. We observe that

$$\operatorname{tr}\left(\frac{\partial \mathbb{D}_j}{\partial x_k}\right) = 0 \ \text{ if } k \le d-1,\ \operatorname{tr}\left(\frac{\partial \mathbb{D}_j}{\partial x_d}\right) = \mathbb{E}_j.$$

For $1 \le j \le d$, we then define

$$\mathbb{Z}_j := (-1)^{d+j}\frac{\partial \mathbb{D}_j}{\partial x_d} - \delta_{jd}\sum_{i=1}^{d}(-1)^{i+d}\frac{\partial \mathbb{D}_i}{\partial x_i} \ \in F_q^{s+1,p}(Q_2) \cap L^\infty(Q_2).$$

Finally, let $\mathbb{Z} := (\mathbb{Z}_1,\dots,\mathbb{Z}_d)$. The identities div $\mathbb{Z} = 0$ and tr $\mathbb{Z} =$ tr $\mathbb{Y}$ are straightforward. The vector field $\mathbb{X} = \mathbb{Y} - \mathbb{Z}$ has all the required properties. □

iii. Inversion of the divergence in epigraphs. This is achieved in Lemma 7.5.
Let $\psi \in C^\infty(\mathbb{R}^{d-1})$ and for $r > 0$,

$$U_r := \{(x',x_d) \in Q'_r \times \mathbb{R} : \psi(x') < x_d < \psi(x') + 1\}.$$

Lemma 7.4 *Let $k = [s]$. There exists $\varepsilon_0 > 0$ such that if $\sum_{i=1}^{k+2} \|D^i\psi\|_{L^\infty(\mathbb{R}^{d-1})} \le \varepsilon_0$, then given any $f \in F_q^{s,p}(U_1)$, we may find some $\mathbb{X} \in F_q^{s+1,p}(U_1) \cap L^\infty(U_1)$ satisfying div $\mathbb{X} = f$ in U_1 and tr $\mathbb{X} = 0$ on $\{(x',\psi(x')) : x' \in Q'_1\}$. Moreover, we may choose $\mathbb{X}$ such that*

$$\|\mathbb{X}\|_{F_q^{s+1,p}(U_1)} + \|\mathbb{X}\|_{L^\infty(U_1)} \le C\|f\|_{F_q^{s,p}(U_1)},$$

with $C > 0$ independent of f.

Proof As in the proof of [5, Lemma 6], for $x' \in Q'_1$ and $0 < y < 1$, set $\tilde{f}(x',y) = f(x',y+\psi(x'))$. When $s \le 0$, this definition has to be understood as $\langle \tilde{f},\varphi\rangle := \langle f,\varphi\circ\Psi^{-1}\rangle$, where $\Psi(x',y) := (x',y+\psi(x'))$. We claim that

$$\|\tilde{f}\|_{F_q^{s,p}(Q_1)} \le c\|f\|_{F_q^{s,p}(U_1)}. \tag{7.5}$$

Indeed, when s is an integer (with $s \ge -1$) and $q = 2$, we have $F_2^{s,p}(\Omega) = W^{s,p}(\Omega)$, and in this case (7.5) is easily checked using the definition of Sobolev spaces. By complex interpolation [24, Theorem 2.13] one obtains (7.5) for all $s \ge -1$ and $q = 2$. Then real interpolation yields $\|\tilde{f}\|_{B_q^{s,p}(Q_1)} \le c\|f\|_{B_q^{s,p}(U_1)}$ for all $s > -1$ and all $q \in \mathbb{R}$. In particular, (7.5) holds for all s whenever $q = p$. Since (7.5) also holds when $q = 2$, complex interpolation yields (7.5) for all s,p and $q \in [2,p]$.

By Step 2, there exist $C_0 > 0$ and $\tilde{\mathbb{X}} \in F_q^{s+1,p}(Q_2) \cap L^\infty(Q_2)$ such that div $\tilde{\mathbb{X}} = \tilde{f}$ in Q_1, tr $\tilde{\mathbb{X}} = 0$ on Q'_1 and

$$\|\tilde{\mathbb{X}}\|_{F_q^{s+1,p}(Q_2)} + \|\tilde{\mathbb{X}}\|_{L^\infty(Q_2)} \le C_0\|\tilde{f}\|_{F_q^{s,p}(Q_1)}.$$

Now, set $\mathbb{X}(x',y) := \tilde{\mathbb{X}}(x',y-\psi(x'))$ and write the components of $\tilde{\mathbb{X}}$ as $\tilde{\mathbb{X}} = (\tilde{\mathbb{X}}',\tilde{\mathbb{X}}^d)$. We obtain

$$\operatorname{div}\mathbb{X}(x',x_d) - f(x',x_d) = \sum_{i=1}^{d-1}\frac{\partial\tilde{\mathbb{X}}^d}{\partial x'_i}(x',x_d-\psi(x'))\frac{\partial\psi}{\partial x'_i}(x').$$

Clearly, this $\mathbb{X}$ satisfies

$$\|\operatorname{div}\mathbb{X} - f\|_{F_q^{s,p}(U_1)} \le C_0\sum_{i=1}^{d-1}\left\|\frac{\partial\tilde{\mathbb{X}}^d}{\partial x'_i}\frac{\partial\psi}{\partial x'_i}\right\|_{F_q^{s,p}(Q_1)}.$$

By [21, Theorem 4.6.2.2] (this requires $s > -1/2$), we obtain

$$\|\operatorname{div}\mathbb{X} - f\|_{F_q^{s,p}(U_1)} \le C_1\left(\|\nabla\psi\|_{L^\infty(\mathbb{R}^{d-1})} + \|\nabla\psi\|_{F_\infty^{s+1,p}(\mathbb{R}^{d-1})}\right)\|f\|_{F_q^{s,p}(U_1)} \le C_2\varepsilon_0\|f\|_{F_q^{s,p}(U_1)}, \tag{7.6}$$

$$\|\mathbb{X}\|_{F_q^{s+1,p}(U_1)} + \|\mathbb{X}\|_{L^\infty(U_1)} \le C_3\|f\|_{F_q^{s,p}(U_1)}, \tag{7.7}$$

where the constants C_i, $0 \le i \le 3$, do not depend on ψ. If we choose $\varepsilon_0 < \dfrac{1}{2C_2}$, then Lemma 7.5 follows by iterating the above construction of $\mathbb{X}$ (as in the proof of Theorem 1.1) and using (7.6) and (7.7). □

We next remove the smallness condition on ψ.

Lemma 7.5 *Let $\psi \in C_c^\infty(\mathbb{R}^{d-1})$. There exists $\delta > 0$ which only depends on d and ψ with the following property: one can find $\mathbb{X} \in F_q^{s+1,p}(U_\delta) \cap L^\infty(U_\delta)$ satisfying div $\mathbb{X} = f$ in U_δ and tr $\mathbb{X} = 0$ on $\{(x', \psi(x')) : x' \in Q'_\delta\}$. Moreover, there exists $C > 0$ (which may depend on ψ but not on f) such that*

$$\|\mathbb{X}\|_{F_q^{s+1,p}(U_\delta)} + \|\mathbb{X}\|_{L^\infty(U_\delta)} \le C\,\|f\|_{F_q^{s,p}(U_1)}.$$

Proof As in the proof of [5, Lemma 7], define $\psi_\delta(x') := \psi(\delta x')$ for any $\delta > 0$. We also define $f_\delta(x', x_d) = f(\delta x', x_d)$. For small δ, we have $\sum_{i=1}^{k+2} \|D^i \psi_\delta\|_{L^\infty(\mathbb{R}^{d-1})} \le \varepsilon_0$, where ε_0 is given by Lemma 7.4. We apply Lemma 7.4 to ψ_δ and f_δ on U_1. This gives a vector field $\mathbb{X}_\delta \in F_q^{s+1,p}(U_1) \cap L^\infty(U_1)$ satisfying div $\mathbb{X} = f_\delta$ on U_1 and tr $\mathbb{X}_\delta = 0$ on $\{(x', \psi_\delta(x')) : x' \in Q'_1\}$ with the corresponding estimates. We then consider for $(x', x_d) \in U_\delta$,

$$\mathbb{X}(x', x_d) = \left(\delta \mathbb{X}'_\delta\left(\frac{x'}{\delta}, x_d\right), \mathbb{X}^d_\delta\left(\frac{x'}{\delta}, x_d\right)\right).$$

We can easily check that $\mathbb{X}$ satisfies all the required properties. □

iv. Proof of Theorem 1.3 completed: inversion of the divergence in general domains.

Lemma 7.6 *There exists a map $\mathbb{S} : F_q^{s,p}(\Omega) \to L^\infty(\Omega) \cap F_q^{s+1,p}(\Omega)$ such that for every $f \in F_q^{s,p}(\Omega)$ we have*

$$\|\mathbb{S}f\|_{F_q^{s+1,p}(\Omega)} + \|\mathbb{S}f\|_{L^\infty(\Omega)} \le C\,\|f\|_{F_q^{s,p}(\Omega)},$$

$$\|f - div\ \mathbb{S}f\|_{F_q^{s+1,p}(\Omega)} \le C\,\|f\|_{F_q^{s,p}(\Omega)},$$

$$tr\ \mathbb{S}f = 0 \text{ on } \partial\Omega,\ tr\ (f - div\ \mathbb{S}f) = 0 \text{ on } \partial\Omega.$$

Proof There exist $\delta > 0$ and a finite covering of $\partial\Omega$ by open sets $\{U_1, \dots, U_l\}$ such that for each $1 \le i \le l$, there exists $\psi_i \in C^\infty(\overline{Q'_\delta})$ satisfying

1. $U_i \cap \Omega$ is isometric to $\{(x', x_d) \in Q'_\delta \times \mathbb{R} : \psi_i(x') < x_d < \psi_i(x') + \delta\}$.
2. $U_i \cap \partial\Omega$ is isometric to $\{(x', x_d) \in Q'_\delta \times \mathbb{R} : \psi_i(x') = x_d\}$.

We may further assume that δ is sufficiently small for the conclusion of Lemma 7.5 to hold true in each U_i.

We then complete the proof using a partition of unity, as in the proof of [5, Lemma 5]. □

In order to complete the proof of Theorem 1.3, we apply the following lemma [5, Lemma 8].

Lemma 7.7 *Let E, F be two Banach spaces and let T be a bounded linear operator from E into F such that Ker $T^* = \{0\}$. Assume that there exists a bounded operator $\mathbb{S}$ from F to E and a compact operator K from F into itself such that $T \circ \mathbb{S} = I + K$. Then T admits a right inverse.*

Proof of Theorem 1.3 completed. We apply the above lemma with

$$E = \{\mathbb{X} \in F_q^{s+1,p}(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d);\ \text{tr}\ \mathbb{X} = 0 \text{ on } \partial\Omega\},$$

$$F = \left\{f \in F_q^{s,p}(\Omega) : \int_\Omega f = 0\right\}$$

and T = div. Then $T^* = -\nabla$, defined on $F' = (F_q^{s,p}(\Omega))'/\mathbb{R}$. If $D \in \mathscr{D}'(\Omega)$ satisfies $\nabla D = 0$, then D is constant on Ω (here, we use the connectedness of Ω). This implies that Ker $T^* = \{0\}$. The existence of $\mathbb{S}$ follows from Lemma 7.6. To define K, we consider the compact embedding $J : F_q^{s+1,p}(\Omega) \to F_q^{s,p}(\Omega)$. Let $K : F \to F$ be the composition of the continuous map $F \ni f \mapsto -f +$ div $\mathbb{S}f \in F_q^{s+1,p}$ with J. Then K is compact and satisfies $T \circ \mathbb{S} = I + K$. We complete the proof of Theorem 1.3 via Lemma 7.7. □

Remark 7.8 1. It is very likely that Theorem 1.3 is true in Lipschitz domains.

2. A similar argument shows that, for all $s > 0$, all $1 < p, q < +\infty$, for all $f \in F_q^{s,p}(\Omega)$ with $\int_\Omega f = 0$, there exists $\mathbb{Y} \in F_q^{s+1,p}(\Omega)$ such that div $\mathbb{Y} = f$, tr $\mathbb{Y} = 0$ on $\partial\Omega$ and $\|\mathbb{Y}\|_{F_q^{s+1,p}(\Omega)} \lesssim \|f\|_{F_q^{s,p}(\Omega)}$. This is a particular case of [15, Theorem 1.1]; see also [7, Theorem 1.1]. However, note that the arguments in [7, 15] are of different nature.

8 Appendix. Some estimates involving trigonometric polynomials

We gather here some elementary results involving trigonometric sums required in the proof of Theorem 1.1; some of them are well-known.

The first result is essentially due to Bernstein, but seems difficult to find in the literature.

Lemma 8.1 *The Fejèr and de la Vallée Poussin kernels satisfy*

$$\|\mathbb{F}_N^{(j)}\|_{L^p} \le C_j(N+1)^{j+1-1/p},\ \|\mathbb{V}_N^{(j)}\|_{L^p} \le C_j(N+1)^{j+1-1/p},\ \forall\, j\in\mathbb{N}, \forall\, 1\le p\le\infty. \tag{8.1}$$

Proof The standard Bernstein's inequality [4, Theorem 11.1.1] asserts that

$$\|f^{(j)}\|_{L^\infty(\mathbb{T})} \le N^j\|f\|_{L^\infty(\mathbb{T})}, \text{ if supp } \widehat{f}\subset [\![-N,N]\!] \text{ and } j\ge 1. \tag{8.2}$$

The integral Bernstein inequality [9, Theorem D.2.1] is

$$\int_{\mathbb{T}} |f(x+a)-f(x)|\,dx \le CN|a|\|f\|_{L^1(\mathbb{T})}, \text{ if supp } \widehat{f}\subset [\![-N,N]\!]. \tag{8.3}$$

The next step is to generalize (8.3) to

$$\|f(\cdot+a)-f\|_{L^p(\mathbb{T})} \le CN|a|\|f\|_{L^p(\mathbb{T})}, \text{ if supp } \widehat{f}\subset [\![-N,N]\!], \forall\, 1\le p\le\infty. \tag{8.4}$$

This is obtained as follows: we have $\widehat{\mathbb{V}_N}(n)=1$ if $|n|\le N$. Therefore, we have $f*\mathbb{V}_N=f$ and $[f(\cdot+a)]*\mathbb{V}_N=f(\cdot+a)$, and thus

$$\|f(\cdot+a)-f\|_{L^p(\mathbb{T})} = \|[f(\cdot+a)-f]*\mathbb{V}_N\|_{L^p(\mathbb{T})} = \|f*[\mathbb{V}_N(\cdot-a)-\mathbb{V}_N]\|_{L^p(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})}\|\mathbb{V}_N(\cdot-a)-\mathbb{V}_N\|_{L^1(\mathbb{T})} \le CN|a|\|f\|_{L^p(\mathbb{T})};$$

here, we used (8.3) (for $\mathbb{V}_N$) and the fact that $\|\mathbb{V}_N\|_{L^1(\mathbb{T})} = \|2\mathbb{F}_{2N}-\mathbb{F}_N\|_{L^1(\mathbb{T})}\le 3$.
By letting $a\to 0$ in (8.4), we find that

$$\|f'\|_{L^p(\mathbb{T})} \le CN\|f\|_{L^p(\mathbb{T})}, \text{ if supp } \widehat{f}\subset [\![-N,N]\!], \forall\, 1\le p\le\infty. \tag{8.5}$$

If we iterate (8.5), then we obtain

$$\|f^{(j)}\|_{L^p(\mathbb{T})} \le C_jN^j\|f\|_{L^p(\mathbb{T})}, \text{ if supp } \widehat{f}\subset [\![-N,N]\!], \forall\, j\in\mathbb{N}^*, \forall\, 1\le p\le\infty. \tag{8.6}$$

On the other hand, we have

$$\|\mathbb{F}_N\|_{L^1}=1,\ \|\mathbb{F}_N\|_{L^\infty}\le 2N+1,\ \mathbb{V}_N = 2\mathbb{F}_{2N}-\mathbb{F}_N,$$

so that

$$\|\mathbb{F}_N\|_{L^p} \le C(N+1)^{1-1/p},\ \|\mathbb{V}_N\|_{L^p}\le C(N+1)^{1-1/p},\ \forall\, 1\le p\le\infty. \tag{8.7}$$

We conclude by combining (8.6) with (8.7). □

The next estimates are known as Nikolskii's inequalities [17]. However, they were known before [17]; see the historical discussion in [16].

Lemma 8.2 *Let $1\le p\le q\le\infty$, $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d$ and $(N_1,\dots,N_d)\in\mathbb{N}^d$. Assume that supp $\widehat{f}\subset\prod_{j=1}^d[\![-N_j,N_j]\!]$. Then*

$$\|\partial^\alpha f\|_{L^q(\mathbb{T}^d)} \lesssim \prod_{j=1}^d (N_j+1)^{\alpha_j+1/p-1/q}\|f\|_{L^p(\mathbb{T}^d)}. \tag{8.8}$$

Proof The identity $f=f*(\mathbb{V}_{N_1}\otimes\ldots\otimes\mathbb{V}_{N_d})$ implies

$$\|\partial^\alpha f\|_{L^q(\mathbb{T}^d)} = \left\|f*\left(\partial_1^{\alpha_1}\mathbb{V}_{N_1}\otimes\ldots\otimes\partial_1^{\alpha_d}\mathbb{V}_{N_d}\right)\right\|_{L^q(\mathbb{T}^d)} \le \|f\|_{L^p(\mathbb{T}^d)}\prod_{j=1}^d\left\|\partial_1^{\alpha_j}\mathbb{V}_{N_j}\right\|_{L^r(\mathbb{T})},$$

where $\dfrac{1}{q}+1=\dfrac{1}{p}+\dfrac{1}{r}$. We conclude via (8.1). □

Corollary 8.3 *Let $1\le p\le q\le\infty$. Assume that supp $\widehat{f}\subset\prod_{j=1}^d[\![a_j,b_j]\!]$. Then*

$$\|f\|_{L^q(\mathbb{T}^d)} \lesssim \prod_{j=1}^d (b_j-a_j+1)^{1/p-1/q}\|f\|_{L^p(\mathbb{T}^d)}. \tag{8.9}$$

Proof Apply Lemma 8.2 to $x\mapsto f(x)e^{-\imath(a_1x_1+\ldots+a_dx_d)}$. □

We next present an estimate involving Fejèr's kernel $\mathbb{F}_N$.

Lemma 8.4 *If* supp $\widehat{f}\subset[\![-N,N]\!]$, *then we have*

$$|f|\le 3|f|*\mathbb{F}_N. \tag{8.10}$$

Proof The starting point is again the identity $f = f * \mathbb{V}_N$, which yields

$$|f| = |f * \mathbb{V}_N| = |f * (2\mathbb{F}_{2N} - \mathbb{F}_N)| \le |f| * |2\mathbb{F}_{2N} - \mathbb{F}_N|.$$

A simple computation shows that $\mathbb{F}_{2N} \le 2\mathbb{F}_N$, so that $|2\mathbb{F}_{2N} - \mathbb{F}_N| \le 3\mathbb{F}_N$, and the proof of (8.10) is complete. □

Corollary 8.5 *Assume that* supp $\widehat{f} \subset \prod_{j=1}^{d} [\![a_j, a_j + N_j - 1]\!]$. *Then*

$$|f| \le 3^d |f| * (\mathbb{F}_{N_1} \otimes \ldots \otimes \mathbb{F}_{N_d}). \tag{8.11}$$

Finally, we present a proof of (5.9).

Lemma 8.6 *Let* $f \in \mathscr{P}(\mathbb{T})$ *be such that* supp $\hat{f} \subset [\![a, b-1]\!]$. *Then*

$$\left||f|'\right| \le (b-a)\left|\mathbb{F}_{b-a} * (e^{-\imath a x} f)\right|. \tag{8.12}$$

Proof We have

$$\begin{aligned} \left||f|'\right| &= \left|\left|\sum_{n=a}^{b-1} \widehat{f}(n) e^{\imath n x}\right|'\right| = \left|\left|\sum_{n=a}^{b-1} \widehat{f}(n) e^{\imath (n-b) x}\right|'\right| \le \left|\sum_{n=a}^{b-1} \widehat{f}(n) \left(e^{\imath (n-b) x}\right)'\right| \\ &= \left|\sum_{n=a}^{b-1} (b-n) \widehat{f}(n) e^{\imath (n-b) x}\right| = \left|\sum_{n=a}^{b-1} (b-n) \widehat{f}(n) e^{\imath (n-a) x}\right| = \left|(b-a)\mathbb{F}_{b-a} * (e^{-\imath a x} f)\right|, \end{aligned}$$

the last equality being easily checked. □

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